

**A Comparative Study of
Oscillatory Integral, and
Sub-Level Set, Operator
Norm Estimates**

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Michael Władisław Kowalski)

*To my darling wife Iona, and to my dear parents who
have sacrificed so much for me.*

Abstract

Oscillatory integral operators have been of interest to both mathematicians and physicists ever since the emergence of the work *Théorie Analytique de la Chaleur* of Joseph Fourier in 1822, in which his chief concern was to give a mathematical account of the diffusion of heat. For example, oscillatory integrals naturally arise when one studies the behaviour at infinity of the Fourier transform of a Borel measure that is supported on a certain hypersurface. One reduces the study of such a problem to that of having to obtain estimates on oscillatory integrals of the following form

$$I(\lambda) = \int e^{i\lambda\Phi(x)}\psi(x)dx$$

where $|\lambda| > 1$, ψ is C^∞ smooth and compactly supported on some sufficiently small set, and Φ is a smooth real-valued *phase function*.

However, sub-level set operators have only come to the fore at the end of the 20th Century, where it has been discovered that the decay rates of the oscillatory integral $I(\lambda)$ above may be obtainable once the measure of the associated *sub-level sets* $\{t \in \text{supp } \psi : |\Phi(t)| < \delta\}$, where $\delta > 0$, are known. This discovery has been fully developed in a paper of A. Carbery, M. Christ and J. Wright [2].

A principal goal of this thesis is to explore certain uniformity issues arising in the study of sub-level set estimates. For example, starting with a sub-level set esti-

mate, or more generally a norm estimate for the *multilinear sub-level set operator*

$$S_{\delta}^{\Phi, K, \pi}(f_1 \dots f_L) = \int_{\{x \in K : |\Phi(x)| < \delta\}} \prod_{j=1}^L f_j(\pi_j(x)) dx,$$

where $K \subset \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L}$ is a compact set, $\pi_j : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}^{m_j}$ are general mappings for each $j \in \{1, \dots, L\}$ belonging to the set $\pi = \{\pi_1, \dots, \pi_L\}$, and $\Phi : K \rightarrow \mathbf{R}$, we ask what further information is needed to guarantee that *uniform* estimates hold when the phase Φ is replaced by $P(\Phi)$ where P is a general polynomial. One would like estimates which are uniform over all polynomials of bounded degree, and we will obtain positive results given an a priori estimate for the associated *multilinear oscillatory integral operator* defined by

$$\Lambda_{\lambda}^{\Phi, K, \pi}(f_1 \dots f_L) = \int_K e^{i\lambda\Phi(x)} \prod_{j=1}^L f_j(\pi_j(x)) dx.$$

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Preliminaries and Notation

The set of all complex numbers will be denoted by \mathbf{C} , the set of all real numbers by \mathbf{R} , the set of all natural numbers $\{1, 2, 3, \dots\}$ by \mathbf{N} , the set of all integers by \mathbf{Z} , the set of all rational numbers by \mathbf{Q} . The set $\mathbf{N} \cup \{0\}$ will be denoted as \mathbf{N}_0 .

For positive A and B , $A \lesssim B$ will mean $A \leq CB$ where C is an absolute constant which may depend on the dimension of the space in question, or have other dependencies in addition to this. Any dependence in a constant that we wish to emphasise will be made clear via subscripts or will be stated explicitly. Automatically, $B \gtrsim A$ means $A \lesssim B$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

If E is a subset of \mathbf{R}^n for $n \geq 1$, $|E|$ will denote its Lebesgue measure and χ_E its characteristic function: $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.

If S is any finite set of elements i.e. $S = \{s_1, \dots, s_d\}$, then $\#S = d$.

Big O and Little o notation

In analysis one frequently requires to prove that a quantity q is “small”, but what it means to be “small” depends on the context: when we stand in a football pitch we do not normally think of its area as being small, but it is indeed small when compared with the area of Great Britain. So, more precisely, one requires in fact

to prove that q is small by comparison with another quantity Q . Furthermore, we are always interested in the *ultimate* comparison between the quantities, and so to say that q is small compared with Q is to say that in the limit, the ratio q/Q is zero. This is precisely where O and o notation, which we will shortly endeavour to define and explain, makes life so much easier, since it allows us to grasp the essence of analytic arguments without having to go through a lot of unnecessary technical details.

For $f, g : \mathbf{R} \rightarrow \mathbf{R}$ (although f and g can be complex valued as well)

$f(h) = O(g(h))$ as $h \rightarrow \infty$ means that there is a positive constant K such that $|f(h)| \leq K|g(h)|$ for all sufficiently large $|h|$;

$f(h) = O(g(h))$ as $h \rightarrow 0$ means that there is a positive constant K such that $|f(h)| \leq K|g(h)|$ for all sufficiently small $|h|$;

$f(h) = o(g(h))$ as $h \rightarrow \infty$ means that $\lim_{|h| \rightarrow \infty} f(h)/g(h) = 0$;

$f(h) = o(g(h))$ as $h \rightarrow 0$ means that $\lim_{|h| \rightarrow 0} f(h)/g(h) = 0$.

Note. We can use the notation in a very flexible way, we can write, for example, $O(g)$ for any function f with the property that, for $K > 0$, that $|f(h)| \leq K|g(h)|$ for sufficiently small (or sufficiently large) $|h|$.

Multi-indices and Schwartz functions

When dealing with \mathbf{R}^n for $n \geq 2$ we will use *multi-indices*, which generalise the concept of an integer index to an array of indices. An n -dimensional *multi-index* is a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{N}_0^n$.

If $\alpha \in \mathbf{N}_0^n$ and $f : \mathbf{R}^n \rightarrow \mathbf{C}$. Then we will simply write

$$\partial_x^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and we will write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

A function $\psi : \mathbf{R}^n \rightarrow \mathbf{C}$ belongs to the *Schwartz class*, $\mathcal{S}(\mathbf{R}^n)$, if ψ is infinitely differentiable and, for all $\alpha, \beta \in \mathbf{N}_0^n$,

$$p_{\alpha, \beta}(\psi) := \sup_{x \in \mathbf{R}^n} |x^\alpha \partial_x^\beta \psi(x)| < \infty.$$

The Schwartz class is a Fréchet space, dense in $L^p(\mathbf{R}^n)$ for all $p \in [1, \infty)$ under the following topology given by the semi-norms $p_{\alpha, \beta}$: a sequence $(\psi_k)_{k \geq 1}$ converges to the zero function if and only if $p_{\alpha, \beta}(\psi_k)$ tends to zero as k tends to infinity, for all $\alpha, \beta \in \mathbf{N}_0^n$. The space $C_c^\infty(\mathbf{R}^n)$ is the space of infinitely differentiable functions of compact support. One thinks of $\mathcal{S}(\mathbf{R}^n)$ as the space consisting of all those functions in $C^\infty(\mathbf{R}^n)$ which decrease rapidly at infinity. More precisely, a Schwartz function is one such that it, together with all its derivatives, decreases more rapidly than any polynomial. The space of *tempered distributions*, $\mathcal{S}'(\mathbf{R}^n)$, is the space of bounded linear functionals on $\mathcal{S}(\mathbf{R}^n)$.

In general, we shall take smooth to mean C^∞ smooth, but when the need arises the degree of smoothness will be made clear. We will denote by $\phi^{(k)}$ the k 'th derivative of ϕ .

L^p Spaces

For a fixed measure space (X, μ) and $p \in [1, \infty]$, $L^p(X)$ denotes the Banach space of those measurable functions $f : X \rightarrow \mathbf{C}$ such that $\|f\|_{L^p}$ is finite, where

$$\|f\|_{L^p} := \left(\int_X |f|^p d\mu \right)^{1/p},$$

with the agreement that when p is ∞ we interpret the above expression as the essential supremum of f on X . When there is no danger of confusion, we simply write L^p for $L^p(X)$. Each $p \in [1, \infty]$ has a dual exponent, denoted by p' , which satisfies $1/p + 1/p' = 1$.

The Fourier Transform

Adopting the notation $x \cdot y$ for the standard inner product of elements x and y in \mathbf{R}^n , the Fourier transform of a finite Borel measure μ on \mathbf{R}^n will be defined by

$$\widehat{d\mu}(\xi) := \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} d\mu(x). \quad (0.0.1)$$

If $d\mu(x) = f(x)dx$ with $f \in L^1(\mathbf{R}^n)$ we write

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

Some useful results

(*Minkowski's inequality*). Let (X, μ) and (Y, ν) be σ -finite measure spaces. Then, for all $p \in [1, \infty)$,

$$\left(\int_X \left| \int_Y f(x, y) d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |f(x, y)|^p d\mu(x) \right)^{1/p} d\nu(y).$$

(*Hölder's inequality*). Let (X, μ) be a σ -finite measure space. Then, for all

$$p \in [1, \infty],$$

$$\left| \int_X f(x)g(x) \, d\mu(x) \right| \leq \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} \left(\int_X |g(x)|^{p'} \, d\mu(x) \right)^{1/p'}.$$

(*Riesz-Thorin Interpolation*). Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and for $0 < \theta < 1$ define p and q by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Suppose T is a linear operator from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$ such that for each $i \in \{0, 1\}$ we have

$$\|Tf\|_{L^{q_i}(\mathbf{R}^n)} \leq C_i \|f\|_{L^{p_i}(\mathbf{R}^n)} \quad \text{for each } f \in L^{p_i}(\mathbf{R}^n),$$

then

$$\|Tf\|_{L^q(\mathbf{R}^n)} \leq C_0^{1-\theta} C_1^\theta \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for each } f \in L^p(\mathbf{R}^n).$$

Two immediate corollaries of Riesz-Thorin Interpolation are the following:

(*Hausdorff-Young inequality*). Assume $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(\mathbf{R}^n)$, then $\hat{f} \in L^{p'}(\mathbf{R}^n)$ and

$$\|\hat{f}\|_{L^{p'}(\mathbf{R}^n)} \leq \|f\|_{L^p(\mathbf{R}^n)}.$$

Proof. It is trivial to observe that $\|\hat{f}\|_{L^\infty(\mathbf{R}^n)} \leq \|f\|_{L^1(\mathbf{R}^n)}$. Moreover, Plancherel's theorem $\|\hat{f}\|_{L^2(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}$ holds for all $f \in \mathcal{S}(\mathbf{R}^n)$. Hence, by continuity the Fourier transform map extends to an isometry of $L^2(\mathbf{R}^n)$, and thus we also have $\|\hat{f}\|_{L^2(\mathbf{R}^n)} \leq \|f\|_{L^2(\mathbf{R}^n)}$. The Hausdorff-Young inequality then follows via interpolation. □

(*Young's inequality*). Assume $1 \leq p, q, r \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. If $f \in L^p(\mathbf{R}^n)$ and $g \in L^q(\mathbf{R}^n)$, then $f * g \in L^r(\mathbf{R}^n)$ and

$$\|f * g\|_{L^r(\mathbf{R}^n)} \leq \|f\|_{L^p(\mathbf{R}^n)} \|g\|_{L^q(\mathbf{R}^n)}.$$

Proof. It is a trivial matter to observe that $\|f * g\|_{L^\infty(\mathbf{R}^n)} \leq \|g\|_{L^\infty(\mathbf{R}^n)} \|f\|_{L^1(\mathbf{R}^n)}$ and that $\|f * g\|_{L^1(\mathbf{R}^n)} \leq \|g\|_{L^1(\mathbf{R}^n)} \|f\|_{L^1(\mathbf{R}^n)}$; Young's inequality then follows via interpolation. □

Πάντες ἄνθρωποι τοῦ εἰδέναι ὀρέγονται φύσει.

ΑΡΙΣΤΟΤΕΛΟΥΣ ΤΩΝ ΜΕΤΑ ΤΑ ΦΥΣΙΚΑ ΒΙΒΛΙΟΝ Α.

All men by nature desire to know.

The Metaphysics, Book A, Aristotle.

Chapter 1

Background and Introduction

In this thesis we shall be concerned with exploring the mapping properties of various sub-level set and oscillatory integral operators, and discussing the relationship that exists between the two. It is well known that, in general, oscillatory integral estimates imply sub-level set estimates, and so the main purpose of the thesis will be to explore this further, and in particular how this relates to stability issues i.e. where one is interested in studying how estimates behave under changes of the phase.

In the preamble that follows, we will set the context of our study in order to show how the oscillatory integral operators that we will encounter arise very naturally from a wide spread of applications and problems. Moreover, we will also elucidate just how naturally interwoven the subject of oscillatory integral operator, and sub-level set measure, estimates is.

1.1 A basic overview of oscillatory integrals

Oscillatory integrals have been the staple diet of problems in harmonic analysis since the very genesis of that subject. In fact, in view of more recent times, one might say that it is a subject which is, in essence, best described as one that

is primarily concerned with investigating the bounding properties of operators that are fashioned from oscillatory integrals arising from a variety of problems. The Fourier transform, which is central to the study of harmonic analysis in \mathbf{R}^n and so vital to the mathematically applied sciences, is itself the most immediate example of such an object. For a suitable function f , its Fourier transform is defined to be

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

However, apart from the obvious example of the Fourier transform, we have a plethora of other instances where these objects make their appearance. The most prominent instances to recall are the occurrence of Bessel functions in the original work of Fourier (1822), the study of asymptotics related to such functions in the early works of Airy (1838), Stokes (1850), and Lipschitz (1859), Riemann's use in 1854 of the method of "stationary phase" in finding the asymptotics of certain Fourier transforms, all of which took place well over 100 years ago.

It is also fascinating to point out, that it is not just harmonic analysis which enjoys the company of these objects. Another impetus for the study of oscillatory integrals, initiated in the first quarter of the 20th Century, came with their application to number theory. In particular, Van der Corput (1922) applied them to studying the distribution of lattice points and their relation to exponential sums. However, given this long history it is an interesting fact that it was only realised relatively recently (1967), that it was possible to obtain restriction theorems for the Fourier transform.

However, the most prominent sphere of application and occurrence of oscillatory integrals, more precisely the Fourier transform, has been in the area of *pseudo-differential operators*. One of the principal motivations for the study of such operators is their wide applicability to *partial differential equations*. Now, almost

all of the physical phenomena that we encounter in the real world which are of any practical or financial interest to man are described by partial differential equations. Hence, oscillatory integrals and the techniques of harmonic analysis are vital tools for understanding how certain things in the real world behave. Now, before we go on any further, it will be convenient to divide our discussion of oscillatory integrals by making a distinction between those of the *first kind* and those of the *second kind*.

1.2 Oscillatory Integrals of the First and Second Kind

Let us return to the Fourier transform itself for a moment and consider the following situation: let S be a smooth m -dimensional submanifold of \mathbf{R}^n , with m in the range $1 \leq m \leq n - 1$. Let $U \subset \mathbf{R}^m$ be a neighbourhood of the origin, and write S as the image of a smooth mapping $\phi : U \rightarrow \mathbf{R}^n$. We think of ϕ as parametrising a m -dimensional surface S in \mathbf{R}^n . We let $d\sigma$ denote the surface measure on S induced by the Lebesgue measure on \mathbf{R}^n , and we fix a function $\psi \in C_0^\infty(\mathbf{R}^n)$ whose support intersects S in a compact subset of S .

Consider now the finite Borel measure $d\mu$ supported in S given by $d\mu(y) = \psi(y)d\sigma(y)$, which is of course carried on S . Let us consider the Fourier transform on S given by

$$\widehat{d\mu}(\xi) = \int_S e^{-2\pi i y \cdot \xi} d\mu(y),$$

our aim is to study its behaviour at infinity, i.e. as $|\xi| \rightarrow \infty$. We proceed as follows, we write

$$\begin{aligned}
\widehat{d\mu}(\xi) &= \int_S e^{-2\pi i y \cdot \xi} d\mu(y) \\
&= \int_{\text{supp } \psi} e^{-2\pi i y \cdot \xi} \psi(y) d\sigma(y) \\
&= \int_{\text{supp } \tilde{\psi}} e^{-2\pi i \phi(x) \cdot \xi} \tilde{\psi}(x) dx \\
&= \int_{\mathbf{R}^m} e^{i|\xi| \Phi(x)} \tilde{\psi}(x) dx
\end{aligned}$$

where $\Phi(x) = -2\pi\phi(x) \cdot \frac{\xi}{|\xi|}$ and $\tilde{\psi} \in C_c^\infty(\mathbf{R}^m)$. Hence, the problem of obtaining estimates on $\widehat{d\mu}$ leads us naturally to the study of *oscillatory integrals of the first kind*, that is, expressions of the form

$$I(\lambda) = \int e^{i\lambda\Phi(x)} \psi(x) dx \tag{1.2.1}$$

where $|\lambda| > 1$, ψ is C^∞ smooth¹ and compactly supported on some sufficiently small set, and Φ is a smooth real-valued *phase function*. In fact, the primary interest in studying these oscillatory integrals is for the purposes of obtaining decay estimates for the Fourier transforms of measures carried on surfaces. Nevertheless, later on in the thesis we will also, in addition to discussing the estimates that have been obtained in the latter half of the 20th Century, discuss how these oscillatory integrals are used to obtain restriction theorems for the Fourier transform.

However, it is appropriate at this point in our discussion on oscillatory integrals to now make a distinction between those of the *first kind* and those of the *second kind*. The main difference being that the former deals with a single function, which can be typically written in the form given in (1.2.1), and the problem is to estimate the decay rates of $I(\lambda)$ for such λ , and to also understand the asymptotic

¹This is the usual situation, but ψ can also be a characteristic function.

behaviour of $I(\lambda)$ as the parameter $\lambda \rightarrow \infty$. One way to accomplish this is to integrate by parts repeatedly, which goes back to Van der Corput, and another method is to develop an asymptotic expansion for $I(\lambda)$, known as the method of “stationary phase”.

In the case of *oscillatory integrals of the second kind*, one is concerned with studying the boundedness properties of an operator that carries an oscillatory factor in its kernel and which can be given in the form

$$T_\lambda(f)(x) = \int e^{i\lambda\Phi(x,y)}\psi(x,y)f(y)dy. \quad (1.2.2)$$

Here, finding estimates for the norm of the operator T_λ as $\lambda \rightarrow \infty$ is the principal goal. While oscillatory integral operators such as the one given in (1.2.2) arise in a variety of forms and have many different uses, we will limit the discussion to only two broad classes of such operators. The first class is more directly derived from the Fourier transform and contains the restriction operators that are connected with oscillatory integrals of the first kind. We will talk about this in more detail in Sections 1.7 and 1.8 of this chapter. Furthermore, the first class also contains the closely connected operators of Bochner-Riesz summability.

A classical problem in Fourier analysis is to make precise the sense in which the Fourier inversion formula

$$f(x) = \int_{\mathbf{R}^n} e^{2\pi i x \cdot \xi} \widehat{f}(\xi) d\xi, \quad \widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx \quad (1.2.3)$$

holds, say for $f \in L^p(\mathbf{R}^n)$. And so a very natural question to ask is whether, for $f \in L^p(\mathbf{R}^n)$, the truncated “partial sums”

$$S_R f(x) = \int_{|\xi| \leq R} e^{2\pi i x \cdot \xi} \widehat{f}(\xi) d\xi$$

converge to f in the L^p norm as $R \rightarrow \infty$.

When $n = 1$ and $1 < p < \infty$, M. Riesz's theorem [37], for the $L^p(\mathbf{R})$ boundedness of the Hilbert transform, gives us a positive answer to the above question. However, when $n \geq 2$ it turns out that we only have L^p convergence in the trivial case $p = 2$. This striking result was proved in the 70's by C. Fefferman [15]. Now, in classical Fourier series one employs summability methods to improve the convergence properties of series, and so a natural way of reformulating the identity in (1.2.3) is in terms of a "summability method"; for instance, one may assert that

$$f(x) = \lim_{R \rightarrow \infty} \int_{\mathbf{R}^n} e^{2\pi i x \cdot \xi} \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\gamma \widehat{f}(\xi) d\xi$$

for suitable γ . One then proceeds to study the bounding properties of the above operator.

Note. $(1 - |x|^2)_+^\gamma$ is defined to be the function which equals $(1 - |x|^2)^\gamma$ when $|x| < 1$ and 0 when $|x| \geq 1$.

The second class of oscillatory integral operators go by the name of *Fourier integral operators* and are an important tool for solving problems arising in partial differential equations and scattering theory. We will provide a more detailed discussion on these operators in the beginning of Chapter 2 and, as the name suggests, we will see that Fourier integral operators have a close connection to the Fourier transform.

Let us return for the moment to the setting of scalar valued oscillatory integrals, namely oscillatory integrals of the form given in (1.2.1). It has been a recent discovery in the latter part of this century that the decay rates of $I(\lambda)$ may be obtainable once the measure of the associated *sub-level sets*

$$\{t \in \text{supp } \psi : |\Phi(t)| \leq \delta\} \quad (\delta > 0)$$

are known. This remarkable discovery has been fully developed in a paper of A. Carbery, M. Christ and J. Wright [4], and, as we mentioned right at the beginning of this chapter, it will be one of the main aims of this thesis to discuss and explore this fascinating interplay further. Moreover, we will see in Chapter 2 that, if one has a priori oscillatory integral estimates available, then one can generally use these oscillatory integral estimates to obtain sub-level set estimates. However, for the time being, we will continue with the subject of scalar valued oscillatory integrals by exploring some of their main features, whilst also expanding a bit more on the relationship that they have with sub-level sets.

1.3 Oscillatory integrals of the first kind and Sub-level set estimates

We begin with the case of one dimension, and shall be concerned with the decay rate estimates of oscillatory integrals of the form

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)dx \tag{1.3.1}$$

where $|\lambda| > 1$, and ϕ is a real-valued C^k smooth function, and ψ is complex valued and C^k smooth.

It should be pointed out that in \mathbf{R} the theory is essentially complete. The principal contributions to $I(\lambda)$ come from the critical points of ϕ . However, the corresponding situation in \mathbf{R}^n is not as straightforward. Here the nature and multiplicity of the critical points becomes more intricate. Nevertheless, one can still establish results about the behaviour of $I(\lambda)$ if the critical point of ϕ is “non-degenerate”. We also have that if some partial derivative is non-vanishing over

the region of integration then $I(\lambda)$ always has a decay of $O(\lambda^{-\epsilon})$, for some $\epsilon > 0$. The latter is the n dimensional analogue of Corollary 1.3.3.

As mentioned earlier, our main motivation for studying oscillatory integrals, more specifically oscillatory integrals of the first kind, is for the purposes of obtaining decay estimates for Fourier transforms of measures carried on surfaces. Here the conditions required on the *phase* ϕ arise from “curvature” conditions on the surfaces under investigation. In this wonderful way oscillatory integrals bring together geometry and harmonic analysis, by providing the link between geometric properties of manifolds and the harmonic analysis related to them. This has two important applications. One is to the study of maximal averages associated with curved surfaces². The other is to restriction theorems for the Fourier transform, a subject which we will look at in more detail in Section 1.7.

Nevertheless, the basic facts about $I(\lambda)$ can be presented in terms of three principles: *localisation*, *scaling*, and *asymptotics*. We will not delve into too much detail on asymptotics in this thesis, but a more thorough exposition on the asymptotic behaviour of $I(\lambda)$ can be found in [41], for a treatment of the single variable case, see pages 334-341; the several variables case can be located on pages 344-347.

1.3.1 Localisation

We observe that the principal contributions to $I(\lambda)$ come from the *critical points* of ϕ , namely those points x such that $\phi'(x) = 0$. We may then, using the localised behaviour of $I(\lambda)$ at these points, develop the total asymptotic behaviour of $I(\lambda)$. The proof of this fact is a consequence of the following:

Proposition 1.3.1. [41] *Let ϕ be smooth, $\psi \in C_c^\infty((a, b))$ and $\phi'(x) \neq 0 \quad \forall x \in$*

²A detailed account of the maximal operator can be found in [40], Chapter 1, §1 and full details on the subject of maximal averages associated with curved surfaces are given in [41], Chapter 11.

$[a, b]$. Then

$$I(\lambda) = O(\lambda^{-N}) \quad \text{as } \lambda \rightarrow \infty$$

for all $N \geq 0$.

Proof. The proof is very straightforward and we simply apply repeated integration by parts to obtain

$$\begin{aligned} \int_a^b e^{i\lambda\phi} \psi dx &= \int_a^b \frac{1}{i\lambda\phi'} \frac{d}{dx} (e^{i\lambda\phi}) \psi dx \\ &= \frac{-1}{i\lambda} \int_a^b e^{i\lambda\phi} \frac{d}{dx} \left(\frac{\psi}{\phi'} \right) dx \\ &= \left(\frac{-1}{i\lambda} \right)^2 \int_a^b e^{i\lambda\phi} \frac{d}{dx} \left(\frac{1}{\phi'} \frac{d}{dx} \left(\frac{\psi}{\phi'} \right) \right) dx \\ &\vdots \\ &= \left(\frac{-1}{i\lambda} \right)^N \int_a^b e^{i\lambda\phi} C_N(x, \phi, \psi) dx \end{aligned}$$

where we have $\forall N \geq 0$

$$|C_N(x, \phi, \psi)| \leq C(\|\phi\|_{C^{N+1}}, \|\psi\|_{C^N}) |\text{supp } \psi| / |\inf \phi'|^{2N} = A_N < \infty,$$

provided $\|\phi\|_{C^{N+1}}$ remains bounded. Thus $|I(\lambda)| \leq A_N \lambda^{-N}$, and the proof is complete. □

This result does extend successfully to higher dimensions via localisation, this is the substance of Proposition 1.5.1, but for the moment we will remain in one dimension.

1.3.2 Scaling

We now state and prove the well known and extremely useful result that goes back to Van der Corput, which exhibits the phenomenon of *scaling*. The result

is as follows:

Proposition 1.3.2. [41] (van der Corput) *Let $\phi : [a, b] \rightarrow \mathbf{R}$ be smooth in $[a, b]$ and suppose that $|\phi^{(k)}(x)| \geq 1 \quad \forall x \in [a, b]$. Then*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-1/k}$$

holds when:

1. $k \geq 2$, or
2. $k = 1$ and $\phi'(x)$ is monotonic.

Before we give the proof of the result we will make a few important comments. Originally, Van der Corput's result was proved by him independently of knowing sub-level set estimates, however, we will take the opportunity in this situation to demonstrate the link between *sub-level set estimates* and *oscillatory integral estimates*, and will thus derive the result via the aid of sub-level set estimates. As a consequence of approaching the proof in this manner we will observe the phenomenon of scaling in the proof of the corresponding sub-level set estimate result instead.

Furthermore, we will only prove part (2) now, and postpone the proof of part (1) until later on. Since we will deduce part (1) as a consequence of Proposition 1.4.1 in Section 1.4 by using sub-level set estimates in conjunction with the result in part (2). Nevertheless, before proving (2) we will briefly outline the scheme of the proof of (1) so as to highlight certain novelties which are pertinent to our discussion so far.

The key idea is as follows, we fix a parameter $\delta \in (0, \infty)$ to be chosen later. We

write

$$\int_a^b e^{i\lambda\phi} = \int_{[a,b] \cap \{|\phi'| \leq \delta\}} e^{i\lambda\phi} + \int_{[a,b] \cap \{|\phi'| > \delta\}} e^{i\lambda\phi} = T_1 + T_2$$

Now $|T_1| \leq |[a, b] \cap \{|\phi'| \leq \delta\}|$, and the term bounding T_1 from above is a *sub-level set*, for which we will obtain the sharp estimate $\delta^{\frac{1}{k-1}}$. The second term T_2 will be taken care of by splitting the region of integration into a disjoint union of subintervals I_j ; in which ϕ' will be monotone and will satisfy the appropriate derivative bounds (up to a *scaling* constant) so that we can invoke (1) on each I_j . We then optimise in δ to give the result.

The main feature of the above proof that one should note is the novel idea of using the sub-level set of ϕ' as a principal ingredient for determining $\int_a^b e^{i\lambda\phi} dx$. In this particular case, the former may be used to obtain information about the latter. Unfortunately this is not always the case in general. However, we will see later on in the thesis that the relationship is reversible, and that we can generally use oscillatory integrals to estimate sub-level sets as well. Hence, sub-level sets and oscillatory integrals should be seen as going hand in hand together.

1.3.3 Proof of part (2) Proposition 1.3.2

Proof. We have

$$\begin{aligned} \int_a^b e^{i\lambda\phi} dx &= \int_a^b \frac{1}{i\lambda\phi'} \frac{d}{dx} e^{i\lambda\phi} dx \\ &= \left[\frac{e^{i\lambda\phi}}{i\lambda\phi'} \right]_a^b - \frac{1}{i\lambda} \int_a^b e^{i\lambda\phi} \frac{d}{dx} \left(\frac{1}{\phi'} \right) dx. \end{aligned}$$

Hence

$$\begin{aligned}
\left| \int_a^b e^{i\lambda\phi} dx \right| &\leq \frac{2}{\lambda} + \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left(\frac{1}{\phi'} \right) \right| dx \\
&= \frac{2}{\lambda} + \frac{1}{\lambda} \left| \int_a^b \frac{d}{dx} \left(\frac{1}{\phi'} \right) dx \right| \quad (\text{by monotonicity of } \phi') \\
&= \frac{2}{\lambda} + \frac{1}{\lambda} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \leq \frac{3}{\lambda}.
\end{aligned}$$

This gives the desired conclusion with $c_1 = 3$. □

Proposition 1.3.2 leads to a similar estimate for integrals of the form (1.3.1), here we do not assume that ψ vanishes near the end points of $[a, b]$.

Corollary 1.3.3. [41] *Under the assumptions on ϕ in Proposition 1.3.2, we have that*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right]$$

Proof. The idea is to use integration by parts to put ourselves in a situation where we can use Proposition 1.3.2. We have

$$\begin{aligned}
\int_a^b e^{i\lambda\phi(x)} \psi(x) dx &= \int_a^b \left(\frac{d}{dx} \int_a^x e^{i\lambda\phi(t)} dt \right) \psi(x) dx \\
&= \left[\int_a^x e^{i\lambda\phi(t)} dt \psi(x) \right]_a^b - \int_a^b \left(\int_a^x e^{i\lambda\phi(t)} dt \right) \psi'(x) dx,
\end{aligned}$$

and using the estimate

$$\left| \int_a^x e^{i\lambda\phi(t)} dt \right| \leq c_k \lambda^{-1/k} \quad \forall x \in [a, b]$$

obtained previously completes the proof. □

1.4 Sub-level set estimates and their relation to van der Corput

We now turn our attention in more detail to sub-level sets and their connection with oscillatory integral estimates. We start with the following well known result (see for example, [41], pages 332-333 or Lemma 3.3 in the paper of M. Christ [7]).

Proposition 1.4.1. *Let $\phi : [a, b] \rightarrow \mathbf{R}$ be smooth and $|\phi^{(k)}(x)| \geq 1 \quad \forall x \in [a, b]$, and for $\delta > 0$ define $E_\delta = \{x \in [a, b] : |\phi(x)| \leq \delta\}$. Then*

$$|E_\delta| \leq c_k \delta^{1/k}$$

holds for each $k \geq 1$.

Proof. The result is proved by induction on k , for the case $k = 1$ we see that $c_k = 2$ as follows: If $E_\delta \neq \emptyset$, then, since $E_\delta \subset [a, b]$, we have that $\exists \alpha_\delta, \beta_\delta \in \mathbf{R}$ such that $\alpha_\delta = \inf E_\delta$ and $\beta_\delta = \sup E_\delta$, so $E_\delta \subset [\alpha_\delta, \beta_\delta]$. It follows that $|\phi(\alpha_\delta)| \leq \delta$ and $|\phi(\beta_\delta)| \leq \delta$ from the smoothness of ϕ and the definition of supremum and infimum. Therefore

$$|E_\delta| \leq |\beta_\delta - \alpha_\delta| = \left| \frac{\phi(\beta_\delta) - \phi(\alpha_\delta)}{\beta_\delta - \alpha_\delta} \right|^{-1} |\phi(\beta_\delta) - \phi(\alpha_\delta)| = |\phi'(\xi)|^{-1} |\phi(\beta_\delta) - \phi(\alpha_\delta)|$$

for some $\xi \in (\alpha_\delta, \beta_\delta)$ by the mean value theorem, and $|\phi'(\xi)| \geq 1 \Rightarrow |E_\delta| \leq 2\delta$.

We now proceed by induction on k . Let us assume that the case k is known and assume (replacing ϕ by $-\phi$ if necessary) that $\phi^{(k+1)}(x) \geq 1 \quad \forall x \in [a, b]$. Then $\phi^{(k)}(x)$ is increasing, let c be the unique point in $[a, b]$ where $|\phi^{(k)}(x)|$ assumes its minimum value. If $\phi^{(k)}(c) = 0$ then, $\forall \epsilon > 0$ and $\forall x \notin (c - \epsilon, c + \epsilon)$, we have that $|\phi^{(k)}(x)| \geq \epsilon$. Write

$$[a, b] = [a, c - \epsilon] \cup (c - \epsilon, c + \epsilon) \cup [c + \epsilon, b] = I^a \cup (c - \epsilon, c + \epsilon) \cup I^b.$$

Hence

$$\begin{aligned} E_\delta &= \{x \in I^a : |\phi(x)| < \delta\} \cup (c - \epsilon, c + \epsilon) \cup \{x \in I^b : |\phi(x)| < \delta\} \\ &= E_\delta^a \cup (c - \epsilon, c + \epsilon) \cup E_\delta^b. \end{aligned}$$

Now $\forall x \in I^s$ we have that $|(\frac{\phi}{\epsilon})^{(k)}(x)| \geq 1$, for $s = a, b$. Hence, by invoking the induction hypothesis, we have that for $s = a, b$

$$|E_\delta^s| \leq c_k(\delta/\epsilon)^{1/k}.$$

Therefore

$$|E_\delta| \leq 2\epsilon + 2c_k(\delta/\epsilon)^{1/k}.$$

If $\phi^{(k)}(c) \neq 0$, and so c is one of the end points of $[a, b]$, a similar argument shows that $\epsilon + c_k(\delta/\epsilon)^{1/k}$ is an upper bound for $|E_\delta|$. In either situation, the case $k + 1$ follows by taking

$$\epsilon = \delta^{1/k+1},$$

which completes the proof with $c_{k+1} = 2c_k + 2$; since $c_1 = 2$, we have $c_k = 2(2^k - 1)$. \square

Remark. The sharpness of the decay rate is observed by taking $[a, b] = [0, 1]$ and $\phi(x) = \frac{x^k}{k!}$.

We will now apply Proposition 1.4.1 to establish part (1) of Proposition 1.3.2 with c_k given explicitly as $6(k - 1) + 2(2^{k-1} - 1)$.

1.4.1 Proof of part (1) Proposition 1.3.2

Proof. Let $k \geq 2$ and $\delta > 0$. We have by assumption $|\phi^{(k)}(x)| \geq 1 \quad \forall x \in [a, b]$. This implies ϕ' has at most $k - 1$ zeros and $k - 2$ local extreme points. Hence

$$\{x \in [a, b] : |\phi'(x)| > \delta\} = \bigcup_{j=1}^{2(k-1)} I_j$$

where the intervals I_j are disjoint and ϕ' is monotone on each, and so we split $[a, b]$ as follows

$$[a, b] = \{x \in [a, b] : |\phi'(x)| \leq \delta\} \cup \left(\bigcup_{j=1}^{2(k-1)} I_j \right).$$

Hence

$$\int_a^b e^{i\lambda\phi(x)} dx = \int_{\{x \in [a, b] : |\phi'(x)| \leq \delta\}} e^{i\lambda\phi(x)} dx + \int_{\bigcup_{j=1}^{2(k-1)} I_j} e^{i\lambda\phi(x)} dx = T_1 + T_2.$$

Note that $|\phi^{(k)}(x)| \geq 1 \quad \forall x \in [a, b]$ is equivalent to $|\phi^{(k-1)}(x)| \geq 1 \quad \forall x \in [a, b]$, so we use the sublevel estimate of Proposition 1.4.1 to estimate T_1 as follows

$$|T_1| \leq |\{x \in [a, b] : |\phi'(x)| \leq \delta\}| \leq c_{k-1} \delta^{\frac{1}{k-1}}.$$

T_2 is taken care of by the following observation: from the earlier discussion we have for each subinterval I_j and $\forall x \in I_j$ that $\phi'(x)$ is monotone and $|(\frac{\phi'}{\delta})(x)| \geq 1$.

Hence we can use the result for $k = 1$ to give

$$\left| \int_{I_j} e^{i\lambda\phi(x)} dx \right| \leq \frac{3}{\lambda\delta}, \quad \text{for } j = 1, 2, \dots, 2(k-1).$$

Upon which applying this to T_2 we obtain

$$|T_2| \leq \frac{6(k-1)}{\lambda\delta}.$$

Hence, combining our two estimates, we see that

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_{k-1} \delta^{\frac{1}{k-1}} + \frac{6(k-1)}{\lambda\delta}.$$

Choosing δ to be equal to $\lambda^{-\left(\frac{k-1}{k}\right)}$ we are led to

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq (6(k-1) + c_{k-1}) \lambda^{-1/k} = (6(k-1) + 2(2^{k-1} - 1)) \lambda^{-1/k}.$$

□

Remark. It is interesting to point out that there is yet another sublevel estimate approach to obtaining van der Corput's result using the following slightly modified sublevel set $E_m(\delta) = \{x \in [a, b] : |\phi^{(m)}(x)| \leq \delta\}$. We give a sketch of the proof. Once again induction is employed, and the case $m = 1$ is dealt with in exactly the same way as in the proof of part (2) in Proposition 1.3.2. In exactly the same manner as the case $k = 1$ was proved in Proposition 1.4.1 we can deduce that if $m \geq 1$ and $\phi^{(m+1)}(x) \geq 1 \quad \forall x \in [a, b]$ then $|E_m(\delta)| \leq 2\delta$. Write

$$\int_a^b = \int_{E_m(\delta)} + \int_{cE_m(\delta)}.$$

The inductive step shows that

$$\left| \int_{cE_m(\delta)} e^{i\lambda\phi(x)} dx \right| \leq 2c_m / (\lambda\delta)^{1/m}.$$

Also note that when $m = 1$, $\phi'' \geq 1$ which implies ϕ' is monotonic. Thus

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq 2\delta + 2c_m / (\lambda\delta)^{1/m}.$$

Choosing $\delta = \lambda^{-\left(\frac{1}{m+1}\right)}$ gives the estimate $2(c_m + 1)\lambda^{-\left(\frac{1}{m+1}\right)}$, the rest of the details regarding the constant c_m follow as before.

1.5 Oscillatory integrals, several variables and submanifolds of finite type

We now turn our attention to \mathbf{R}^n for $n \geq 2$ where unfortunately only some of the one dimensional results for oscillatory integrals of the first kind have analogues. From now onwards we will be dealing with *multi-indices*. In several variables, the extension of Proposition 1.3.2 is simple, however the extension of Corollary 1.3.3 is less satisfactory and only a weak analogue of it can be asserted.

Nevertheless, continuing in the terminology of the one dimensional case, we say that a *phase function* ϕ defined in a neighbourhood of a point x_0 has x_0 as a *critical point* if

$$(\nabla\phi)(x_0) = \left(\frac{\partial\phi}{\partial x_1}, \dots, \frac{\partial\phi}{\partial x_n} \right) \Big|_{x=x_0} = 0.$$

Moreover, in addition to this, if the determinant of the symmetric $n \times n$ matrix

$$\left[\frac{\partial^2\phi}{\partial x_i \partial x_j} \right] (x_0)$$

is nonzero, then the critical point x_0 is said to be *non-degenerate*.

We have the following n dimensional analogue of Proposition 1.3.1.

Proposition 1.5.1. [41] *Suppose $\psi \in C_c^\infty(\mathbf{R}^n)$, ϕ is a smooth real-valued function and that $\nabla\phi \neq 0$ on the support of ψ . Then*

$$I(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda\phi(x)} \psi(x) dx = O(\lambda^{-N}),$$

as $\lambda \rightarrow \infty$, for every $N \geq 0$.

Proof. The strategy is to reduce matters via *localisation* to a situation where we can apply the one-dimensional result in Proposition 1.3.1. That is, we wish to put

ourselves in the situation where we are able to assume without loss of generality that in some direction x_m , with $x' \in \mathbf{R}^{n-1}$, we have that $\left| \frac{\partial}{\partial x_m} \phi(x_m, x') \right| \geq C > 0$; once this has been accomplished we can then apply the one-dimensional localisation result in this direction.

We proceed as follows, we have $\forall t \in \text{supp } \psi$ that $|\nabla \phi(t)| \geq c_t > 0$. Fix a t_0 belonging to $\text{supp } \psi$, and consider an arbitrary $\xi \in \mathbf{R}^n$; for this ξ we have

$$\begin{aligned} |\xi \cdot (\nabla \phi)(x)| &= |\xi \cdot (\nabla \phi)(t_0) + \xi \cdot [(\nabla \phi)(x) - (\nabla \phi)(t_0)]| \\ &\geq |\xi \cdot (\nabla \phi)(t_0)| - |\xi \cdot [(\nabla \phi)(x) - (\nabla \phi)(t_0)]|. \end{aligned}$$

Choosing ξ now to be a unit vector, and applying the mean value theorem, we obtain

$$|\xi \cdot [(\nabla \phi)(x) - (\nabla \phi)(t_0)]| \leq \|\phi\|_{C^2} |x - t_0|.$$

Hence, if we choose $\xi = \frac{\nabla \phi(t_0)}{|\nabla \phi(t_0)|}$, and $|x - t_0| < r_{t_0}$ where $r_{t_0} = \frac{c_{t_0}}{2\|\phi\|_{C^2}}$, it then follows that $|\xi \cdot (\nabla \phi)(x)| \geq \frac{1}{2}c_{t_0} > 0$; and so, we have shown that $\forall t \in \text{supp } \psi$, there exists a unit vector ξ and a small ball $B_{r_t}(t)$, centred at t , so that

$$|\xi \cdot (\nabla \phi)(x)| \geq c_t/2 > 0$$

for all $x \in B_{r_t}(t)$, with $r_t = \frac{c_t}{2\|\phi\|_{C^2}}$, provided $\|\phi\|_{C^2}$ is bounded of course.

The consequence of this is that we have a collection of balls $\{B_{r_t}\}_{t \in \text{supp } \psi}$ that cover $\text{supp } \psi$. However, since $\text{supp } \psi$ is compact, there exist indices k with $k = 1, \dots, M$ where

$$\text{supp } \psi \subseteq \bigcup_{k=1}^M B_{r_{t_k}}(t_k).$$

Let $c = \min_{1 \leq k \leq M} c_{t_k}$, then $c > 0$, since $M < \infty$ and $c_{t_k} > 0 \ \forall k : 1 \leq k \leq M$.

Hence,

$$\forall x \in B_{r_{t_i}}(t_i) \text{ we have that } |\xi \cdot (\nabla \phi)(x)| \geq c/2.$$

So the support of ψ is covered by a finite collection of balls with the above property.

We now decompose the integral $I(\lambda)$ as a finite sum

$$\sum_k \int_{B_k} e^{i\lambda\phi(x)} \psi_k(x) dx,$$

via a partition of unity of balls $B_k = B_{r_{t_k}}(t_k)$ with radius $c_{t_k}/2\|\phi\|_{C^2}$, where each ψ_k is smooth and has compact support in the ball B_k . It then suffices to prove the corresponding estimate for each of these integrals. Consider now the integral

$$\int_{B_k} e^{i\lambda\phi(x)} \psi_k(x) dx,$$

we may assume that for some $\xi \in \mathbf{S}^{n-1}$, we have $|\xi \cdot (\nabla \phi)(x)| \geq c/2$ on B_k , and by a rotation, where we chose a coordinate system x_1, \dots, x_n so that x_m lies along ξ , we may also assume that $\xi = (0, \dots, 1, \dots, 0) = e_m$. Thus, $\left| \frac{\partial \phi}{\partial x_m} \right| \geq c/2$ on B_k .

Now

$$\int_{B_k} e^{i\lambda\phi(x)} \psi_k(x) dx = \int \left(\int e^{i\lambda\phi(x_m, x')} \psi_k(x_m, x') dx_m \right) dx'$$

where $x = (x_m, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$. The inner integral is rapidly decreasing, and is $O(\lambda^{-N})$ for all N by an application of Proposition 1.3.1. Integrating in the remaining variables, and repeating this procedure for all the other M integrals in the sum decomposition of $I(\lambda)$, and finally summing up together all of these contributions gives our final conclusion. \square

The next result is the n -dimensional extension of Corollary 1.3.3, and unfortunately, the one-dimensional result doesn't carry over to n -dimensions as fully as one would hope, as the constant in the n -dimensional estimate depends on the

phase ϕ . However, there is still much work being devoted to answering the question as to whether it is possible to remove the phase function dependency in the estimate.

Proposition 1.5.2. *[41] Suppose ψ is smooth and is supported in the unit ball; also let ϕ be a real valued function so that, for some multi-index α with $|\alpha| > 0$, we have*

$$|\partial_x^\alpha \phi| \geq 1$$

throughout the support of ψ . Then

$$\left| \int_{\mathbf{R}^n} e^{i\lambda\phi(x)} \psi(x) dx \right| \leq c_k(\phi) \cdot (\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}) \cdot \lambda^{-1/k}$$

where $k = |\alpha|$; the constant $c_k(\phi)$ is independent of λ and ψ , and remains bounded as long as the C^{k+1} norm of ϕ remains bounded.

Proof. We begin in exactly same fashion as we did in the previous proof, the strategy being exactly the same as before, namely we do whatever is necessary to reduce matters to the case of one dimension. However, this time our wish is to reduce matters via *scaling* to a situation where we can apply the one-dimensional result in Proposition 1.3.2. That is, we wish to put ourselves in the situation where we are able to assume without loss of generality that in some direction x_m , with $x' \in \mathbf{R}^{n-1}$, we have that $\left| \frac{\partial^k}{\partial x_m^k} \phi(x_m, x') \right| \geq C > 0$. Once this has been accomplished, the one-dimensional van der Corput lemma can then be applied in this direction.

We will again endeavour to find a unit vector ξ that will give us a direction in which the k 'th partial derivative in absolute value is bounded from below. However, before we embark on this initial goal, we state without detailed proof, as the details are not necessary for understanding the overall corpus of the argument presented here, a crucial observation necessary for its achievement. The

interested reader can find more on the details on page 343 of [41].

Consider the real space of homogeneous polynomials of degree k in \mathbf{R}^n ; let $d(k, n)$ denote its dimension. Of course, $\{x^\alpha : |\alpha| = k\}$ is a basis for this space. It can be shown that there are unit vectors

$$\eta^1, \dots, \eta^{d(k, n)}$$

so that the homogeneous polynomials

$$(\eta^j \cdot x)^k, \quad j = 1, \dots, d(k, n)$$

also give a basis.

Taking this fact for granted, we deduce that for a fixed t_0 in the unit cube that

$$\widehat{\partial_x^\alpha \phi}(t_0) = \left(\sum_{j=1}^{d(k, n)} [(\eta^j \cdot \nabla)^k \phi] \right)^\wedge(t_0).$$

Hence

$$|\partial_x^\alpha \phi(t_0)| = \left| \sum_{j=1}^{d(k, n)} (\eta^j \cdot \nabla)^k \phi(t_0) \right| \leq \sum_{j=1}^{d(k, n)} |(\eta^j \cdot \nabla)^k \phi(t_0)|$$

Let $\xi = \eta^{j_{\max}}$ be the unit vector which gives the largest contribution to the value of $|(\eta^j \cdot \nabla)^k \phi(t_0)|$ then

$$|\partial_x^\alpha \phi(t_0)| \leq \sum_{j=1}^{d(k, n)} |(\eta^j \cdot \nabla)^k \phi(t_0)| \leq d(k, n) |(\xi \cdot \nabla)^k \phi(t_0)|,$$

and since $|\partial_x^\alpha \phi(t_0)| \geq 1$, we have found a unit vector ξ so that

$$|(\xi \cdot \nabla)^k \phi(t_0)| \geq \frac{1}{d(k, n)} = a_k > 0.$$

In a similar way to the previous proposition, we write

$$\begin{aligned} |(\xi \cdot \nabla)^k \phi(x)| &= |(\xi \cdot \nabla)^k \phi(t_0) + (\xi \cdot \nabla)^k \phi(x) - (\xi \cdot \nabla)^k \phi(t_0)| \\ &\geq |(\xi \cdot \nabla)^k \phi(t)| - |(\xi \cdot \nabla)^k \phi(x) - (\xi \cdot \nabla)^k \phi(t_0)|. \end{aligned}$$

Now, using the mean value theorem and the fact that ξ is a unit vector, we conclude that

$$|(\xi \cdot \nabla)^k \phi(x) - (\xi \cdot \nabla)^k \phi(t_0)| \leq A_k \|\phi\|_{C^{k+1}} |x - t_0|.$$

So provided we assume that the C^{k+1} norm of ϕ is bounded, we are then able to choose $|x - t_0| < a_k/2A_k\|\phi\|_{C^{k+1}}$, so that we obtain $|(\xi \cdot \nabla)^k \phi(x)| \geq a_k/2$; and so, we see that for all t in the unit cube that there exists a unit vector ξ , and a ball $B_r(t)$ with a fixed radius $r = O(\|\phi\|_{C^{k+1}}^{-1})$ such that

$$|(\xi \cdot \nabla)^k \phi(x)| \geq a_k/2 \quad \forall x \in B_{r_k}(t).$$

We next choose an appropriate covering $\{B_j\}$ of the unit cube by such balls of fixed radius, of which there are $O(\|\phi\|_{C^{k+1}}^n)$ and a corresponding partition of unity

$$1 = \sum_j \eta_j(x), \quad 0 \leq \eta_j \leq 1, \quad \sum_j |\nabla \eta_j| \leq b_k,$$

with $\text{supp } \eta_j \subset B_j$.

We then write $\psi_j = \psi \cdot \eta_j$ and so

$$\int e^{i\lambda\phi(x)} \psi(x) dx = \sum_j \int_{B_j} e^{i\lambda\phi(x)} \psi_j(x) dx.$$

It then suffices to estimate each $\int_{B_j} e^{i\lambda\phi} \psi_j dx$. So with $\xi \in \mathbf{S}^{n-1}$ determined as above, we have $\forall x \in B_j$ that $|(\xi \cdot \nabla)^k \phi(x)| \geq a_k/2$, and by a rotation where we

choose a coordinate system x_1, \dots, x_n so that x_m lies along ξ , we may also assume that $\xi = (0, \dots, 1, \dots, 0) = e_m$. Thus, $\left| \frac{\partial^k \phi}{\partial x_m^k} \right| \geq a_k/2$ on B_j . Now

$$\int_{B_k} e^{i\lambda\phi(x)} \psi_j(x) dx = \int \left(\int e^{i\lambda\phi(x_m, x')} \psi_j(x_m, x') dx_m \right) dx'$$

where $x = (x_m, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$. For the inner integral we now apply Corollary 1.3.3 giving us an estimate of the form

$$c_k(a_k\lambda)^{-1/k} \left\{ \|\psi_j\|_{L^\infty} + \int \left| \frac{\partial \psi_j}{\partial x_m}(x_m, x') \right| dx_m \right\}.$$

Now

$$\begin{aligned} \left| \frac{\partial \psi_j}{\partial x_m} \right| &\leq |\eta_j| \left| \frac{\partial \psi}{\partial x_m} \right| + |\psi| \left| \frac{\partial \eta_j}{\partial x_m} \right| \\ &\leq \left| \frac{\partial \psi}{\partial x_m} \right| + \|\psi\|_{L^\infty} \left| \frac{\partial \eta_j}{\partial x_m} \right| \\ &\leq \left| \frac{\partial \psi}{\partial x_m} \right| + \|\psi\|_{L^\infty} \sum_j |\nabla \eta_j| \\ &\leq \left| \frac{\partial \psi}{\partial x_m} \right| + b_k \|\psi\|_{L^\infty}, \end{aligned}$$

and also $\|\psi_j\|_{L^\infty} \leq \|\psi\|_{L^\infty}$, so that the estimate for the inner integral is actually of the form

$$c'_k(a_k\lambda)^{-1/k} \left\{ \|\psi\|_{L^\infty} + \int \left| \frac{\partial \psi}{\partial x_m}(x_m, x') \right| dx_m \right\}.$$

Integrating in the other remaining variables, and using the fact that

$$\int \left| \frac{\partial \psi}{\partial x_m}(x) \right| dx \leq \int \left| \frac{\partial \psi}{\partial x_1}(x) \right| + \dots + \left| \frac{\partial \psi}{\partial x_n}(x) \right| dx = \|\nabla \psi\|_{L^1},$$

finally gives us the uniform estimate of

$$d_k \cdot \lambda^{-1/k} \cdot (\|\psi\|_{L^\infty} + \|\nabla \psi\|_{L^1})$$

for each integral $\int_{B_j} e^{i\lambda\phi} \psi_j dx$. Finally, summing up the remaining $O(\|\phi\|_{C^{k+1}}^n)$ contributions completes the estimate of the integral $\int_{\mathbf{R}^n} e^{i\lambda\phi} \psi dx$. We note that the final constant d'_k in our estimate of the above integral depends on the phase ϕ , and remains bounded as long as the C^{k+1} norm of ϕ remains bounded. \square

Now, we have seen with Proposition 1.5.1 that if $\nabla\phi \neq 0$ on the support of ψ , then we get very rapid decay indeed, namely for every $N \geq 0$ we have $I(\lambda) = O(\lambda^{-N})$. However, this should be of no surprise. Since, if ϕ is not constant, then the factor $e^{i\lambda\phi(x)}$ becomes highly oscillatory as $\lambda \rightarrow \infty$, and, for this reason, one expects $I(\lambda)$ to decay rapidly as λ increases without bound. On the other hand, if $\nabla\phi = 0$ somewhere, say at x_0 , then the phase will be stationary at this point, and if in addition to this, the critical point x_0 is non-degenerate i.e. $\det(\frac{\partial^2\phi(x_0)}{\partial x_i \partial x_j}) \neq 0$, it will turn out that

$$I(\lambda) = O(\lambda^{-n/2}).$$

This is known as the phenomenon of *stationary phase*. The proof of this involves the use of *Morse's lemma*, and we simply state the result formally without proof for the sake of completeness, and advise the interested reader to consult pages 344-347 of [41] should they wish to see the details of it.

Proposition 1.5.3. [41] *Suppose $\phi(x_0) = \nabla\phi(x_0) = 0$, and that the critical point x_0 is non-degenerate. If ψ is smooth and supported in a sufficiently small neighbourhood of the critical point x_0 , then*

$$\int_{\mathbf{R}^n} e^{i\lambda\phi(x)} \psi(x) dx = O(\lambda^{-n/2}).$$

Remark. We will refer to this result later on, and it is worth mentioning at this stage that it is a result of great importance. It has a wide range of application, in particular, it is crucial in computing the Fourier transform of measures supported on smooth hypersurfaces which have non-vanishing Gaussian curvature, and this

in turn has vital consequences in the problem of ascertaining the distribution of n -dimensional lattice points. This comment very aptly leads us on to the next section, where we will settle and direct our attention a bit more to the subject of surface carried measures and their Fourier transform estimates.

1.6 Fourier transforms of measures supported on surfaces and submanifolds of finite type

In this section we will continue the discussion that was motivated by the last proposition in the previous section, and we will also briefly set up the background for obtaining a basic restriction theorem for the Fourier transform. Let S be an open subset of a smooth m -dimensional submanifold of \mathbf{R}^n . We let $d\sigma$ denote the measure on S induced by Lebesgue measure on \mathbf{R}^n , and we fix a function $\psi \in C_0^\infty(\mathbf{R}^n)$ whose support intersects S in a compact subset of S .

For a finite measure μ on \mathbf{R}^n , we define its Fourier transform by

$$\widehat{d\mu}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} d\mu(x).$$

Note. If $d\mu(x) = f(x)dx$ with $f \in L^1(\mathbf{R}^n)$ then the above becomes the ordinary Fourier transform on $L^1(\mathbf{R}^n)$ and we write $\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x)dx$.

In our future discussion, we will deal with the finite Borel measure $d\mu(y) = \psi(y)d\sigma(y)$ on \mathbf{R}^n , which is carried on S . We will wish to observe the behaviour at infinity of the Fourier transform of $d\mu$, as it will be of use to us in our efforts to obtain a basic restriction theorem for the Fourier transform. Nevertheless, before we go any further in this matter, it is now worth returning to the discussion started by the last proposition of the previous section; and mentioning that this type of problem in fact made its historical debut in number theory. In particu-

lar, it is heavily connected with the study of the distribution of lattice points in regions of \mathbf{R}^n .

However, before we go into more detail, it is crucial to note that when S is the unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$, one can show utilising spherical coordinates and asymptotic expansions of Bessel functions that when $n > 1$ the Fourier transform

$$\widehat{d\sigma}(\xi) = \int_{\mathbf{S}^{n-1}} e^{-2\pi i x \cdot \xi} d\sigma(x)$$

obeys the following decay estimate

$$|\widehat{d\sigma}(\xi)| = O(|\xi|^{(1-n)/2}). \quad (1.6.1)$$

Nevertheless, estimates such as the one above are of a much more general nature, and are completely deducible from the “curvature” properties of the manifold in question, and thus are not limited to the circumstantial luck of being able to connect rotational symmetry with Bessel functions. In fact, it turns out that if we have a smooth hypersurface S in \mathbf{R}^n which has nonzero Gaussian curvature everywhere; then the Fourier transform of the measure $d\mu$ carried on S , obeys exactly the same estimate as given in (1.6.1). This is achieved by invoking the method of stationary phase.

Now the stationary phase method as used in the context above was first applied by Hardy and Littlewood, and Hlawka to the number theoretic problem of ascertaining the distribution of lattice points in regions of \mathbf{R}^n ; and stated in the most simple terms, the result they achieved says the following: If we take the unit ball B in \mathbf{R}^n and let $N(\lambda)$ denote the cardinality of the set of lattice points $j \in \mathbf{Z}^n$ satisfying $|j| \leq \lambda$, then as $\lambda \rightarrow \infty$

$$N(\lambda) = \lambda^n |B| + O(\lambda^{n-2+\frac{2}{n+1}}).$$

The more general n -dimensional result involving a bounded open domain Ω with a smooth boundary still remains largely an unsolved problem. Nevertheless, the case where Ω is assumed to be strictly convex³ was dealt with by Hlawka in 1950 and also by Herz in 1962.

The interested reader is advised to consult pages 49-54 of [39], and §5.12 of [41], for further details and discussion. However, we will now return to the problem of observing the behaviour at infinity of $\widehat{d\mu}$ and move it on towards a more general setting. This, for us, will mean replacing curvature assumptions by the more general notion of a submanifold of *finite type*.

Let us consider now the following situation: let S be a smooth m -dimensional submanifold of \mathbf{R}^n , with m in the range $1 \leq m \leq n-1$. Let $U \subset \mathbf{R}^m$ be a neighbourhood of the origin, and write S as the image of a smooth mapping $\phi : U \rightarrow \mathbf{R}^n$. We think of ϕ as parametrising a m -dimensional surface S in \mathbf{R}^n . If ϕ is such that for a fixed $x_0 \in U$ and for all $\eta \in \mathbf{S}^{n-1}$ there exists a multi index α , with $|\alpha| \geq 1$, such that

$$\partial_x^\alpha [\phi(x) \cdot \eta]|_{x=x_0} \neq 0,$$

then we say ϕ is of *finite type* at x_0 . The smallest k so that, for all $\eta \in \mathbf{S}^{n-1}$, there exists an α with $|\alpha| \leq k$ for which the above holds is defined to be the *type* of ϕ and of S at x_0 . If this is true for all points of S we say that S is a surface of finite type k .

³This means that Ω is convex, bounded and its boundary $\partial\Omega$ has strictly positive Gaussian curvature at each point.

Now if we have a surface S of finite type k , then using Proposition 1.5.2 we can obtain a decay estimate for the Fourier transform of the measure $d\mu = \psi(y)d\sigma$ on \mathbf{R}^n supported on this surface S .

Theorem 1.6.1. [41] *Suppose S is a smooth m -dimensional manifold in \mathbf{R}^n of finite type. Let $d\mu = \psi d\sigma$ be as above. Then*

$$|\widehat{d\mu}(\xi)| \leq A|\xi|^{-1/k},$$

where k is the type of S inside the support of ψ .

Proof. We have that ψ is a function in $C_c^\infty(\mathbf{R}^n)$ whose support intersects S in a compact subset of S , and that k is the type of S inside the support of ψ . Let $U \subset \mathbf{R}^m$ be a neighbourhood of the origin, we also have that S is written as an image of a smooth mapping $\phi : U \rightarrow \mathbf{R}^n$, which we think of as parametrising the m -dimensional surface S in \mathbf{R}^n . The finite Borel measure $d\mu$ is of course carried on S and supported in S , and is given by $d\mu(y) = \psi(y)d\sigma(y)$.

So we can write

$$\begin{aligned} \widehat{d\mu}(\xi) &= \int_S e^{-2\pi i y \cdot \xi} d\mu(y) \\ &= \int_{\text{supp } \psi} e^{-2\pi i y \cdot \xi} \psi(y) d\sigma(y) \\ &= \int_{\text{supp } \tilde{\psi}} e^{-2\pi i \phi(x) \cdot \xi} \tilde{\psi}(x) dx \\ &= \int_{\mathbf{R}^m} e^{i|\xi| \Phi(x)} \tilde{\psi}(x) dx \end{aligned}$$

where $\Phi(x) = -2\pi \phi(x) \cdot \frac{\xi}{|\xi|}$ and $\tilde{\psi} \in C_c^\infty(\mathbf{R}^m)$. Let $\eta = \frac{\xi}{|\xi|}$ so that $\eta \in \mathbf{S}^{n-1}$.

Now, the type k hypothesis implies that we will have for all $t \in \text{supp } \tilde{\psi}$ that

$$\partial_x^\alpha [\phi(x) \cdot \eta]|_{x=t} \neq 0$$

for some α with $|\alpha| \leq k$. Consequently, we have that for all $t \in \text{supp } \tilde{\psi}$

$$|\partial_x^\alpha \Phi(t)| \geq C_t$$

for some α with $|\alpha| \leq k$.

So, by constructing balls of a certain radius centred around each $t \in \text{supp } \tilde{\psi}$, such that $|\partial_x^\alpha \Phi(x)| \geq C_t$ for all x belonging to these balls, we can obtain due to the compactness of $\text{supp } \tilde{\psi}$ for some M a finite sub-covering $\{B_i\}_{i=1}^M$ of $\text{supp } \tilde{\psi}$. We then let $C = \min_{1 \leq i \leq M} C_{t_i}$, so that $|\partial_x^\alpha \Phi(x)| \geq C$ in each ball B_i , and we can use exactly the same style of finite partition of unity technique to estimate the decay rate of $\widehat{d\mu}$.

However, since we are mainly interested in the rate of decay, we are free to take $\text{supp } \tilde{\psi}$ sufficiently small enough so that it is essentially contained in one of these balls, and so we may assume that for some α with $|\alpha| \leq k$ that we have $|\partial_x^\alpha \Phi(x)| \geq C$ for all $x \in \text{supp } \tilde{\psi}$. Hence the conclusion, in terms of the decay,

$$\left| \int_{\mathbf{R}^m} e^{i|\xi|\Phi(x)} \tilde{\psi}(x) dx \right| \leq A_k(\Phi) \cdot A(C) \cdot |\xi|^{-1/k}$$

follows immediately upon the application of Proposition 1.5.2, where $A_k(\Phi)$ is independent of $|\xi|$ and remains bounded so long as the C^{k+1} norm of Φ remains bounded (and since $\Phi(x) = -2\pi\phi(x) \cdot \eta$ with $\eta \in \mathbf{S}^{n-1}$, the constant $A_k(\Phi)$ remains bounded so long as $\|\phi\|_{C^{k+1}}$ remains bounded). Hence, if we would wish to finally add up the estimates on all of the balls in the finite partition of unity we will have

$$|\widehat{d\mu}(\xi)| \leq A|\xi|^{-1/k}$$

where $A = A(\phi)$ is bounded so long as the C^{k+1} norm of ϕ is bounded, and is dependent on the number of balls in the partition of unity.

□

We will use this result in the next section to obtain a restriction theorem for the Fourier transform.

1.7 Restriction of the Fourier transform

Suppose S is a given smooth submanifold of \mathbf{R}^n and that $d\sigma$ is its induced Lebesgue measure. We say that the L^p *restriction property* holds for S if there exists a $q = q(p)$ so that the inequality

$$\left(\int_{S_0} |\widehat{f}(\xi)|^q d\sigma(\xi) \right)^{1/q} \leq A_{p,q}(S_0) \cdot \|f\|_{L^p(\mathbf{R}^n)}$$

holds for each $f \in \mathcal{S}(\mathbf{R}^n)$ whenever S_0 is an open subset of S with compact closure in S . The fact that $\mathcal{S}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$ means we can, whenever the above inequality holds, define \widehat{f} on S (a.e. with respect to $d\sigma$), for each $f \in L^p(\mathbf{R}^n)$.

Before we state and prove the restriction theorem we recall and prove the following important result which will aid us in our proof of it.

(H-L-S) Hardy-Littlewood-Sobolev inequality:

$$\|f * |\cdot|^{-\gamma}\|_{L^q(\mathbf{R}^n)} \leq A_{p,q} \|f\|_{L^p(\mathbf{R}^n)}$$

$$\text{provided } 0 < \gamma < n, 1 < p < q < \infty, \text{ and } 1/q = 1/p - 1 + \gamma/n. \quad (1.7.1)$$

Note. The proof presented here is attributed to Hedberg [1972].

Proof. For $R > 0$ let $B(R) = \{y \in \mathbf{R}^n : |y| < R\}$ and $\Phi(y) = |y|^{-\gamma} \chi_{B(R)}(y)$. We

proceed directly and write

$$\begin{aligned}
(f * |\cdot|^{-\gamma})(x) &= \int_{\mathbf{R}^n} f(x-y)|y|^{-\gamma} dy \\
&= \left(\int_{|y| < R} + \int_{|y| \geq R} \right) f(x-y)|y|^{-\gamma} dy \\
&= (f * \Phi)(x) + \int_{\mathbf{R}^n} f(x-y)|y|^{-\gamma} \chi_{eB(R)}(y) dy \\
&= I_1 + I_2.
\end{aligned}$$

Now I_1 is the convolution of f and a function Φ that is radial, (radially) decreasing and integrable. Recall that if M is the (usual) maximal operator given by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(r)|} \int_{|y|<r} |f(x-y)| dy = \sup_{r>0} \frac{1}{|B(r)|} (|f| * \chi_{B(r)})(x),$$

then whenever Φ is a nonnegative integrable function on \mathbf{R}^n that is radial and (radially) decreasing, we have

$$\sup_{t>0} |(f * \Phi_t)(x)| \leq (Mf)(x) \cdot \|\Phi\|_{L^1(\mathbf{R}^n)}, \quad (1.7.2)$$

where for $t > 0$, define $\Phi_t(x) = t^{-n}\Phi(x/t)$.

To see this, we first prove the result when Φ is a simple function of the form $\sum_{j=1}^N a_j \chi_{B_{r_j}}$ satisfying the above hypotheses, where each $a_j > 0$ and B_{r_j} is the ball of radius r_j centred at the origin. Then

$$|(f * \Phi_t)(x)| \leq \sum_{j=1}^N a_j |B_{r_j}| \frac{1}{|B_{tr_j}|} (|f| * \chi_{B_{tr_j}})(x) \leq \|\Phi\|_{L^1(\mathbf{R}^n)} (Mf)(x)$$

since $\|\Phi\|_{L^1(\mathbf{R}^n)} = \sum_{j=1}^N a_j |B_{r_j}|$. In general, any nonnegative, integrable, radial and (radially) decreasing Φ can be approximated by a monotonically increasing sequence of such finite sums, hence (1.7.2) holds as claimed.

We now apply (1.7.2) to see that I_1 is bounded by

$$(Mf)(x) \cdot \int_{|y| \leq R} |y|^{-\gamma} dy = cR^{n-\gamma} \cdot (Mf)(x).$$

Hölder's inequality implies that I_2 is dominated by

$$\|f\|_{L^p(\mathbf{R}^n)} \cdot \|\cdot\|^{-\gamma} \chi_{cB(R)}\|_{L^{p'}(\mathbf{R}^n)}.$$

Now $\|\cdot\|^{-\gamma} \chi_{cB(R)} \in L^{p'}(\mathbf{R}^n)$ provided $\gamma p' < -n$ and, in view of (1.7.1),

$$\gamma p' - n = \frac{np'}{q} > 0.$$

Thus

$$\|\cdot\|^{-\gamma} \chi_{cB(R)}\|_{L^{p'}(\mathbf{R}^n)} = cR^{-n/q}.$$

Hence, upon combining our two estimates, we see that

$$|(f * \|\cdot\|^{-\gamma})(x)| \leq A[(Mf)(x) \cdot R^{n-\gamma} + \|f\|_{L^p(\mathbf{R}^n)} \cdot R^{-n/q}].$$

Finally, we choose R so that both terms on the right side are equal, i.e.,

$$(Mf)(x)/\|f\|_{L^p(\mathbf{R}^n)} = R^{-n+\gamma-n/q} = R^{-n/p},$$

and substituting this in the above gives

$$|(f * \|\cdot\|^{-\gamma})(x)| \leq A \cdot [(Mf)(x)]^{p/q} \cdot \|f\|_{L^p(\mathbf{R}^n)}^{1-p/q}.$$

The $L^p - L^q$ bound for the operator $f * \|\cdot\|^{-\gamma}$ now follows upon applying the usual $L^p - L^p$ inequality⁴ for M . □

⁴A detailed account of the maximal operator can be found in [40], Chapter 1, §1.

So with all the tools established we can now proceed to demonstrate the restriction property of the Fourier transform.

Theorem 1.7.1. [41] *Suppose S is a smooth m -dimensional submanifold of \mathbf{R}^n of type k . Then there exists a $p_0 = p_0(S) > 1$, so that S has the L^p restriction property with $q = 2$ and $1 \leq p \leq p_0$.*

Note. We will see that

$$p_0 = \frac{2nk}{2nk - 1};$$

and we note that further improvements can be found in §5.14, Chapter 8, and also in §2, Chapter 9, of [41].

Proof. We wish to prove

$$\left(\int_{S_0} |\widehat{f}(\xi)|^2 d\sigma(\xi) \right)^{1/2} \leq A \cdot \|f\|_{L^p(\mathbf{R}^n)}, \quad \text{for } f \in \mathcal{S}(\mathbf{R}^n).$$

Hence it will suffice to prove for $\psi \geq 0$, smooth, compactly supported, such that $\psi = 1$ on S_0 and $\psi = 0$ on $S \setminus S_0$ that

$$\left(\int_S |\widehat{f}(\xi)|^2 \psi(\xi) d\sigma(\xi) \right)^{1/2} \leq A \cdot \|f\|_{L^p(\mathbf{R}^n)}, \quad \text{for } f \in \mathcal{S}(\mathbf{R}^n).$$

Unraveling the left hand side of the above inequality, we have that

$$\left(\int_S |\widehat{f}(\xi)|^2 \psi(\xi) d\sigma(\xi) \right)^{1/2} = \left(\int_S \left| \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx \right|^2 \psi(\xi) d\sigma(\xi) \right)^{1/2}.$$

Let us set $d\mu = \psi d\sigma$ and deal with the operator R , defined for $\xi \in S$ by the Fourier transform

$$(Rf)(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

Consequently the unraveled expression above becomes

$$\left(\int_S |Rf(\xi)|^2 d\mu(\xi) \right)^{1/2},$$

making our interest now divert to considering whether the following is true

$$\|R(f)\|_{L^2(S, d\mu)} \leq A\|f\|_{L^p(\mathbf{R}^n)}.$$

In trying to establish this fact we will work with its formal adjoint R^* , given by

$$(R^*f)(x) = \int_S e^{2\pi i x \cdot \xi} f(\xi) d\mu(\xi) \quad \text{for } x \in \mathbf{R}^n.$$

Using Hölder's inequality we see that

$$\|R(f)\|_{L^2(S, d\mu)}^2 = \langle Rf, Rf \rangle_{L^2(S, d\mu)} = \langle R^*Rf, f \rangle_{L^2(\mathbf{R}^n)} \leq \|R^*R(f)\|_{L^{p'}(\mathbf{R}^n)} \|f\|_{L^p(\mathbf{R}^n)},$$

where p' is the exponent conjugate to p . So to prove

$$R : L^p(\mathbf{R}^n) \rightarrow L^2(S, d\mu)$$

is bounded, it suffices to see that

$$R^*R : L^p(\mathbf{R}^n) \rightarrow L^{p'}(\mathbf{R}^n)$$

is bounded. We have however that

$$(R^*Rf)(x) = \int_{\mathbf{R}^n} \int_S e^{2\pi i \xi \cdot (x-y)} d\mu(\xi) f(y) dy,$$

and so $(R^*Rf)(x) = (f * K)(x)$ with

$$K(x) = \widehat{d\mu}(-x).$$

By Theorem 1.6.1 we have

$$|K(x)| \leq A|x|^{-1/k},$$

also K is trivially bounded since

$$\begin{aligned} |K(x)| &= |\widehat{d\mu}(-x)| = \left| \int_{\mathbf{R}^n} e^{-2\pi i(-x) \cdot \xi} \psi(\xi) d\sigma(\xi) \right| \\ &\leq \int_{\mathbf{R}^n} |\psi(\xi)| d\sigma(\xi) \\ &\leq \|\psi\|_{L^\infty} |S^{n-1}| < \infty, \end{aligned}$$

so

$$|K(x)| \leq A|x|^{-\gamma}, \quad \text{whenever } 0 \leq \gamma \leq 1/k.$$

Hence

$$\|R^*R(f)\|_{L^{p'}(\mathbf{R}^n)} \leq A\|f\| * |\cdot|^{-\gamma} \|_{L^{p'}(\mathbf{R}^n)} \quad \text{for } 0 \leq \gamma \leq 1/k.$$

The H - L - S inequality now leads us to deduce

$$\|R^*R(f)\|_{L^{p'}(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}$$

whenever $1 < p < p' < \infty$, and $1/p' = 1/p - 1 + \gamma/n$. Hence $2n(1 - 1/p) = \gamma$, and the restriction $0 \leq \gamma \leq 1/k$ becomes $1 \leq p \leq (2nk)/(2nk - 1)$, completing the proof of the theorem. □

1.8 Oscillatory integrals of the second kind and restriction

In Section 1.2 we came across, in the context of talking about restriction operators and Bochner-Riesz summability, the notion of oscillatory integrals of the second kind; that is, oscillatory integral operators T_λ defined by

$$(T_\lambda f)(x) = \int e^{i\lambda\Phi(x,y)} \psi(x,y) f(y) dy.$$

Now we have just seen in the previous section the involvement of oscillatory integrals of the first kind in the restriction of the Fourier transform to surfaces of finite type, and so, it will be our intention now in this section to give a brief survey of how oscillatory integrals of the second kind naturally arise in the restriction of the Fourier transform to hypersurfaces of non vanishing Gaussian curvature.

The following restriction result for a hypersurface S (i.e., a submanifold of dimension $n - 1$) whose Gaussian curvature does not vanish anywhere can be found in Chapter 9 of [41], we simply state the result and refer the reader to pages 387-388 should they wish to see the details of the proof.

Proposition 1.8.1. *[41] Let $S \subset \mathbf{R}^n$ be a manifold of dimension $n - 1$ whose Gaussian curvature is nowhere zero, and let S_0 be a compact subset of S . Then*

$$\left(\int_{S_0} |\widehat{f}(\xi)|^q d\sigma(\xi) \right)^{1/q} \leq A(S_0) \cdot \|f\|_{L^p(\mathbf{R}^n)}$$

holds for each $f \in \mathcal{S}(\mathbf{R}^n)$, whenever $1 \leq p \leq \frac{2n+2}{n+3}$ and $q = \left(\frac{n-1}{n+1}\right)p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Note. If $p = \frac{2n+2}{n+3}$ then $q = 2$, while $p = 1$ gives $q = \infty$.

Even though we omit the details of the proof, we will mention, however, certain key points from it that pertain to the general discussion as to how it is that oscillatory integrals of the second kind arise in the study of Fourier restriction. The strategy behind the proof of the above result is to localise to a small neighbourhood of a fixed point in S_0 , and to apply an appropriate change of variables (moving the point to the origin) so that one may assume that near that point the hypersurface S is given as the graph

$$x_n = \phi(x_1, \dots, x_{n-1}) = \phi(x')$$

where the phase ϕ satisfies a suitable nondegeneracy condition.

One then, via a suitable nonnegative cut-off function $\tilde{\psi} \in C_0^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^n)$, reduces matters to studying the bounding properties of the oscillatory integral operator of the second kind given by

$$(T_\lambda^* f)(x') = \int_{\mathbf{R}^n} e^{-i\lambda\Phi(x', \xi)} \tilde{\psi}(x', \xi) f(\xi) d\xi, \quad (1.8.1)$$

where the modified phase

$$\Phi(x', \xi) = 2\pi(x' \cdot \xi' + \phi(x')\xi_n),$$

with $\xi = (\xi', \xi_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$, also satisfies a suitable nondegeneracy condition.

The mapping T_λ^* maps functions on \mathbf{R}^n to functions on \mathbf{R}^{n-1} , and is the adjoint operator of the operator T_λ , which maps functions on \mathbf{R}^{n-1} to functions on \mathbf{R}^n , given by

$$(T_\lambda f)(\xi) = \int_{\mathbf{R}^{n-1}} e^{i\lambda\Phi(x', \xi)} \psi(x', \xi) f(x') dx'. \quad (1.8.2)$$

Here $\psi \in C_0^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^n)$ is a cut-off function, and the phase $\Phi(x', \xi)$ is the same phase as given before in the operator T_λ^* . To complete the restriction result, one must obtain the appropriate $L^p - L^q$ mapping properties for the operator given in (1.8.1). To achieve this, one employs the dual formulation of the following result, which is given for the operator in (1.8.2).

Theorem 1.8.2. *[41] Under certain assumptions of nondegeneracy on the phase Φ , the operator given in (1.8.2) satisfies the estimate*

$$\|T_\lambda f\|_{L^q(\mathbf{R}^n)} \leq A\lambda^{-n/q} \|f\|_{L^p(\mathbf{R}^{n-1})},$$

where

$$q = \left(\frac{n+1}{n-1} \right) p' \quad \text{and} \quad 1 \leq p \leq 2;$$

here $p' = (1 - p^{-1})^{-1}$ is the exponent conjugate to p .

We state it without proof, and we refer the reader to Chapter 9 of [41], pages 380-386, should they wish to see the details. Furthermore, we also direct the reader to [41], pages 379-380, if they also wish to understand the nondegeneracy conditions assigned to the phase Φ , since we have obviously omitted their description in the statement of the theorem.

We note in passing that the adjoint operator T_λ^* in (1.8.1), which is mapping functions on \mathbf{R}^{n-1} to functions on \mathbf{R}^n , is a generalisation of the operator R^* that arose in the previous section. Moreover, Theorem 1.8.2, via duality, also gives us an estimate on the bilinear form given by

$$\Lambda(f, g) = \int e^{i\lambda\Phi(x,y)} \psi(x, y) f(x) g(y) dx dy, \quad (1.8.3)$$

and thus enables us to know its boundedness as well.

Now, another oscillatory integral of the second kind T_λ , that naturally arises and may be thought of as a generalisation of the Fourier transform, is one that maps functions from \mathbf{R}^n to functions on \mathbf{R}^n and is given by

$$(T_\lambda f)(x) = \int_{\mathbf{R}^n} e^{i\lambda\Phi(x,y)} \psi(x, y) f(y) dy, \quad (1.8.4)$$

where ψ is a fixed smooth function of compact support in x and y (a “cut-off” function); and Φ is real valued and in $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. We assume that, $\det(\frac{\partial^2 \Phi(x,y)}{\partial x_i \partial y_j}) \neq 0$ on the support of ψ . The following result concerning the L^2 bounding properties of the operator in (1.8.4) above, will be used later on in

certain calculations of sub-level set operator norms. We simply state the result and refer the reader to [41], Chapter 9, §1, pages 377-379, should they wish to see the details of the proof.

Proposition 1.8.3. [41] (Hörmander) *Under the above assumption on Φ and ψ , we have that*

$$\|T_\lambda(f)\|_{L^2(\mathbf{R}^n)} \leq A\lambda^{-n/2}\|f\|_{L^2(\mathbf{R}^n)}. \quad (1.8.5)$$

Remark. If we consider the special case when $\Phi(x, \xi)$ is bilinear and non-degenerate, namely when $\Phi(x, \xi) = -2\pi x \cdot \xi$, and also the family of operators

$$(\tilde{T}_\lambda f)(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} \psi(x/\lambda^{1/2}, \xi/\lambda^{1/2}) f(x) dx,$$

then after we have re-scaled and applied proposition 1.8.3 we find that

$$\|\tilde{T}_\lambda(f)\|_{L^2(\mathbf{R}^n)} \leq A\lambda^{-n/2} \left(\int_{\mathbf{R}^n} |f(\lambda^{1/2}x)|^2 dx \right)^{1/2} \lambda^{n/2} \lambda^{n/4} = A\|f\|_{L^2(\mathbf{R}^n)}.$$

If we now choose ψ so that $\psi(0, 0) = 1$ and let $\lambda \rightarrow \infty$, we reduce matters to the Fourier transform and obtain $\|\hat{f}\|_{L^2(\mathbf{R}^n)} \leq A\|f\|_{L^2(\mathbf{R}^n)}$, which is Plancherel's theorem.

It is very easy to see that $\|T_\lambda\|_{L^1 \rightarrow L^\infty} \leq C$, and so we can apply Riesz-Thorin interpolation to obtain the following result.

Corollary 1.8.4. *If $1 \leq p \leq 2$, then*

$$\|T_\lambda\|_{L^p \rightarrow L^q} \leq A\lambda^{-n/q},$$

where q denotes the conjugate exponent.

1.9 Overview of thesis and main results

The bulk of this thesis, which, as the reader will discover, is contained in Chapter 3, is devoted to the exploration of certain uniformity issues arising in the study of sub-level set bounds.

In Chapter 2 we explore a bit further the relationship that exists between sub-level set estimates and oscillatory integral estimates of the first kind in the context of oscillatory integrals of the second kind. We see how the study of sub-level set operators also arises naturally from the study Fourier integral operators. In Section 2.3, for technical purposes, we establish Proposition 2.3.2. This proposition then enables us to utilise the Fourier inversion formula in order to improve a result of A. Comech and S. Roudenko [11], which is concerned with the L^2 mapping properties of a certain class of sub-level set operator connected to the study of L^2 bounds of certain Fourier integral operators. We refer to Theorem 2.3.3 for details.

Finally, we bring Chapter 2 to a close by establishing some further sub-level set results. In Section 2.4.1 we utilise Morse's lemma to prove a local sub-level set result, and in Section 2.4.2 we prove that if we assume certain a priori asymptotics on oscillatory integrals, then we are able to obtain lower bounds on sub-level set estimates.

In Chapter 3 we study the stability of sub-level set estimates in a multilinear setting. More precisely, we consider the *multilinear sub-level set operator*

$$S_{\delta}^{\Phi, K, \pi}(f_1 \dots f_L) = \int_{\{x \in K : |\Phi(x)| < \delta\}} \prod_{j=1}^L f_j(\pi_j(x)) dx,$$

where $K \subset \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L}$ is a compact set, $\pi_j : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}^{m_j}$ are

general mappings for each $j \in \{1, \dots, L\}$ belonging to the set $\pi = \{\pi_1, \dots, \pi_L\}$, and $\Phi : K \rightarrow \mathbf{R}$.

Assuming a priori estimates for the associated *multilinear oscillatory integral operator* defined by

$$\Lambda_{\lambda}^{\Phi, K, \pi}(f_1 \dots f_L) = \int_K e^{i\lambda\Phi(x)} \prod_{j=1}^L f_j(\pi_j(x)) dx,$$

our objective is to ascertain what further information is needed to guarantee that *uniform* estimates hold when the phase Φ is replaced by $P(\Phi)$ where P is a *normalised* polynomial.

We are naturally led to analyse the finer structure of sub-level sets

$$S_{\delta, P} := \{t \in \mathbf{R} : |P(t)| < \delta\},$$

and to obtain bounds for $|S_{\delta, P}|$ that are *global* in t . These bounds in fact follow as a consequence of the following stronger result, Theorem 3.3.7, which we prove in Section 3.4. More precisely, Theorem 3.3.7 says that for any $k \geq 0$ there exists $d(k)$ such that for any normalised real polynomial $P(t) = \sum_{m=0}^d c_m t^m$ with $d \geq d(k)$ and $|c_{d-k}| = 1$, there is an absolute constant $A = A_d$ so that one has the uniform set inclusion

$$\{t \in \mathbf{R} : |P(t)| < \delta\} \subset \bigcup_{\xi \in \mathcal{R}_P} \{t \in \mathbf{R} : |t - \Re(\xi)| \leq A_d \delta^{1/d}\}$$

for all $0 \leq \delta < 1$. Here \mathcal{R}_P denotes the set of roots of P .

We obtain the proof of the above result by exploiting the well known algebraic fact that the coefficients of any polynomial P can be described in terms of the elementary symmetric polynomials of its roots, and our analysis utilises a basic

decomposition of S. Dendrinos and J. Wright [12] along with some algebra to optimise estimates. Furthermore, the reader will discover that we will end up proving a slightly more general variant of Theorem 3.3.7.

In Section 3.6, we then apply Theorem 3.3.7 to the setting of multilinear sub-level set operator estimates and prove Theorem 3.6.1, from which immediately follows Corollary 3.6.2. We conclude the chapter, with Section 3.7, where we obtain a stability result for sub-level set operators. We refer to Proposition 3.7.1 for details.

Finally, in Chapter 4 we study the oscillatory integral $\int e^{i\lambda\Phi(x)}\psi(x)dx$ in the setting of asymptotic expansions and explore the possibility of obtaining a *calculus of oscillatory integral estimates* in one dimension, for the particular simple example where the derivatives of the phase Φ , at the critical point x_0 , satisfy

$$\Phi'(x_0) = \dots = \Phi^{(k-1)}(x_0) = 0,$$

while $\Phi^{(k)}(x_0) \neq 0$ with $k \geq 2$.

It is well known, that the above conditions on the phase Φ allow us to obtain the much stronger full *asymptotic expansion*, in terms of powers of λ , for the oscillatory integral $\int e^{i\lambda\Phi(x)}\psi(x)dx$, where ψ is a smooth function supported in a sufficiently small neighbourhood of x_0 .

The main aim of this chapter is to identify the conditions under which a phase of the form $\Psi(x) = P(\Phi(x))$, where P and Φ are real-valued and smooth, satisfies $\Psi'(x_0) = \dots = \Psi^{(\ell-1)}(x_0) = 0$ but $\Psi^{(\ell)}(x_0) \neq 0$. This then gives us the necessary conditions for establishing a calculus of oscillatory integral estimates in the setting of asymptotic expansions. We use the formula of Faà di Bruno to prove that provided P satisfies

$$P'(\Phi(x_0)) = \dots = P^{(n-1)}(\Phi(x_0)) = 0, \quad P^{(n)}(\Phi(x_0)) \neq 0,$$

for $n \geq 1$, then $\Psi(x) = P(\Phi(x))$ satisfies

$$\Psi'(x_0) = \dots = \Psi^{(nk-1)}(x_0) = 0, \quad \Psi^{(nk)}(x_0) \neq 0.$$

This then allows us to obtain

$$\int e^{i\lambda P(\Phi(x))} \psi(x) dx \sim \lambda^{-1/nk} \sum_{j=0}^{\infty} a_j \lambda^{-j/nk},$$

where the asymptotics hold in the same sense as in (4.2.1). We refer to Theorem 4.2.3 for details. We then conclude the chapter with a direct application of the above result to case when $P(t) = t^n$, $n \geq 2$, is a monomial function. We refer to Corollary 4.2.4 for details.

Chapter 2

Oscillatory integrals and sub-level set estimates

2.1 Introduction

In the previous chapter we studied examples of how both oscillatory integrals of the first and second kind naturally arise. Having also observed the relationship that exists between sub-level set estimates and oscillatory integral estimates of the first kind, we will in this chapter endeavour to explore this aspect a bit further in the context of oscillatory integrals of the second kind. We will consider the relation between sub-level set operators and Fourier integral operators, and we will see how the study of sub-level set operators also arises naturally from the study Fourier integral operators. We will examine the contents of the paper [11] by A. Comech and S. Roudenko which are pertinent to our study of sub-level set operator estimates and we will improve a certain result contained therein. Finally, we will bring the chapter to a close by establishing some further sub-level set results.

2.2 Fourier integral operators and sub-level set operators

Fourier integral operators have in the last 25 years become an important tool in the study of partial differential equations and scattering theory. While several generalisations are possible, in its basic form, a *Fourier integral operator* T is given by

$$(Tf)(x) = \int_{\mathbf{R}^n} e^{2\pi i \Phi(x, \xi)} a(x, \xi) \widehat{f}(\xi) d\xi$$

where the function a has compact support in x , and we assume that it is a symbol of standard type; namely, that it belongs to one of the classes S^m , where S^m is defined to be the standard class of all symbols a that satisfy the following condition:

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|},$$

for all multi-indices α and β , with $a(x, \xi)$ being a C^∞ function of $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$. The fixed number m is called the order of the symbol. The phase Φ is real-valued, homogeneous of degree 1 in ξ , and smooth in (x, ξ) , for $\xi \neq 0$, on the support of a . We also assume that Φ satisfies the crucial non-degeneracy condition that, for $\xi \neq 0$,

$$\det \left(\frac{\partial^2 \Phi}{\partial x_i \partial \xi_j} \right) \neq 0$$

on the support of a .

The simplest example of a Fourier integral operator is when $\Phi(x, \xi) = x \cdot \xi$. In that case when $a(x, \xi) = 1$ for large ξ , we essentially get the identity operator, which expresses the Fourier inversion formula. Indeed, it is this example that is primarily responsible for the genesis of the name *Fourier integral operator*.

For our purposes, we shall consider the Fourier integral operator, with $n \geq 1$,

$$(\mathcal{F}u)(x) = \int_{\mathbf{R}^n \times \mathbf{R}^n} e^{i\phi(x, \theta, y)} a(x, \theta, y) u(y) d\theta dy,$$

where the phase function $\phi(x, \theta, y)$ is smooth and homogeneous of degree 1 in the variable θ , and

$$a \in S_{\text{cl}}^d(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$$

is a classical symbol of order d , introduced by Hörmander. For simplicity, we assume that $a(x, \theta, y)$ has compact support in both x and y and is homogeneous in θ (of degree d):

$$a(x, \theta, y) = |\theta|^d a(x, \theta/|\theta|, y) \quad \text{for some } d \in \mathbf{R}.$$

The properties of Fourier integral operators are intrinsically related to the properties of sub-level set operators. The latter operators come into play in the following way: let $\varphi \in C_c^\infty(\mathbf{R})$, where $\text{supp } \varphi = [0, 1]$ and φ is identically equal to 1 in a neighbourhood of the point $\frac{1}{2}$. Define \mathcal{F}_λ by cutting off large values of θ :

$$(\mathcal{F}_\lambda u)(x) = \int_{\mathbf{R}^n \times \mathbf{R}^n} |\theta|^d e^{i\phi(x, \theta, y)} \varphi(|\theta|/\lambda) a(x, \theta/|\theta|, y) u(y) d\theta dy,$$

let $\omega = \theta/|\theta| \in S^{N-1}$ and $t = |\theta|/\lambda$ so that $\theta = |\theta|\omega = \lambda t\omega$. Then

$$(\mathcal{F}_\lambda u)(x) = \int_{\mathbf{R}^n} \int_0^\infty \int_{S^{N-1}} r^d e^{ir\phi(x, \omega, y)} \varphi(r/\lambda) a(x, \omega, y) r^{N-1} dr d\omega u(y) dy$$

where we have put $r = |\theta|$. Now set $\frac{r}{\lambda} = t$ and so the above integral becomes

$$\begin{aligned} (\mathcal{F}_\lambda u)(x) &= \lambda^{N+d} \int_{\mathbf{R}^n} \int_0^\infty \int_{S^{N-1}} t^{d+N-1} e^{i\lambda t\phi(x, \omega, y)} \varphi(t) a(x, \omega, y) u(y) d\omega dt dy \\ &= \lambda^{N+d} \int_{S^{N-1} \times \mathbf{R}^n} \int_0^\infty t^{d+N-1} \varphi(t) e^{i\lambda t\phi(x, \omega, y)} dt a(x, \omega, y) u(y) d\omega dy. \end{aligned}$$

We write $\psi_0(\tau) = \int_0^\infty t^{d+N-1} \varphi(t) e^{it\tau} dt$, and having chosen our φ as we did allows us to integrate by parts to our heart's content, and hence we have $\psi_0 \in \mathcal{S}(\mathbf{R})$.

So

$$(\mathcal{F}_\lambda u)(x) = \lambda^{N+d} \int_{S^{N-1} \times \mathbf{R}^n} \psi_0(\lambda \phi(x, \omega, y)) a(x, \omega, y) d\omega u(y) dy.$$

If there is no dependence on ω , then

$$(\mathcal{F}_\lambda u)(x) = \lambda^{N+d} \int_{\mathbf{R}^n} \psi_0(\lambda \phi(x, y)) a(x, y) u(y) dy,$$

which is a *sub-level set operator*.

Hence, another way to study Fourier integral operators is to consider integral operators associated to sub-level sets

$$(\mathcal{L}_\lambda u)(x) = \int_{\mathbf{R}^n} \psi(\lambda \phi(x, y)) a(x, y) u(y) dy, \quad \mathcal{L}_\lambda : L^1(\mathbf{R}^n) \longrightarrow L^\infty(\mathbf{R}^n),$$

with $\psi \in \mathcal{S}(\mathbf{R})$.

We take ϕ to be a phase function with $\phi \in \mathcal{C}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and $a \in L_c^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. We will concern ourselves with studying the $L^2 \longrightarrow L^2$ bounding properties of the operator \mathcal{L}_λ , and obtaining decay estimates for large values of λ . In addition, we will also study the sub-level sets themselves that are associated to \mathcal{L}_λ and the measure estimates which they satisfy for large λ .

We now turn our attention to the paper by A. Comech and S. Roudenko [11], where the following result has been obtained for dimension $n = 2$.

Proposition 2.2.1. *Let $n = 2$, $\psi \in \mathcal{S}(\mathbf{R})$, and $a \in L_c^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, that the*

integral operator defined by

$$(\mathcal{L}_\lambda f)(x) = \int_{\mathbf{R}^n} a(x, y) \psi(\lambda \Phi(x, y)) f(y) dy \quad (2.2.1)$$

with $\Phi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, and $\det(\frac{\partial^2 \Phi(x, y)}{\partial x_i \partial y_j}) \neq 0$ on the support of a , obeys the following $L^2 \rightarrow L^2$ estimate:

$$\|\mathcal{L}_\lambda\|_{L^2 \rightarrow L^2} \leq C \lambda^{-1} \log^4 \lambda. \quad (2.2.2)$$

We will demonstrate in the next section that for $n = 2$ the power of 4 in the logarithm can be improved to 1. We shall also conclude that our result is sharp.

2.3 Obtaining sub-level set estimates from oscillatory integral estimates

In this section we will discuss the main connection between oscillatory integral operator estimates and sub-level set estimates. There is an abundance of sharp results for oscillatory integrals available, both of the first and second kind. We will see via a well known technique, that is found in harmonic analysis folklore, how to get back a type of sub-level set estimate when we have an a priori oscillatory integral operator decay estimate.

The sub-level set operator S_δ , associated to the sub-level set E_δ given by

$$E_\delta = \{(x, y) \in U \times V : |\Phi(x, y)| < \delta\}$$

where U and V are compact sets in \mathbf{R}^n , that we will first consider for the moment, is the most basic and is given by

$$(S_\delta f)(x) = \int_{\mathbf{R}^n} \chi_{E_\delta}(x, y) f(y) dy. \quad (2.3.1)$$

One can readily see that once we know the mapping properties of the sub-level set operator, we can deduce, as a simple consequence of choosing the appropriate characteristic functions, the measure of the sub-level set. The folklore technique will translate the mapping properties of the oscillatory integral operator to that of the sub-level set operator. We will also show later on in the thesis, that if one has a priori asymptotics on an oscillatory integral, then one can adapt this technique to also obtain lower bounds for the corresponding sub-level set measure.

We have already encountered in Chapter 1 many examples of oscillatory integrals of the second kind that arise from various problems. For example, in section 1.8 we saw these objects naturally arise in the study of the restriction of the Fourier transform to hypersurfaces of non vanishing Gaussian curvature. The following theorem manifests the folklore technique, and demonstrates the connection between oscillatory integrals of the second kind and sub-level set estimates:

Theorem 2.3.1. *Let S_δ be defined as above in (2.3.1), and consider*

$$(T_\lambda f)(x) = \int_{\mathbf{R}^n} e^{i\lambda\Phi(x,y)} \psi(x,y) f(y) dy,$$

where ψ is any nonnegative function which is compactly supported in both variables x and y , and is identically one on $U \times V$, and suppose that

$$\|T_\lambda\|_{L^p \rightarrow L^q} \leq C|\lambda|^{-\eta}$$

holds for some $\eta \geq 0$ with some constant C independent of λ . Then

$$\|S_\delta\|_{L^p \rightarrow L^q} \leq AC \begin{cases} \delta^\eta & 0 \leq \eta < 1, \\ \delta \log\left(\frac{1}{\delta}\right) & \eta = 1, \\ \delta & \eta > 1, \end{cases}$$

where A is an absolute constant.

Proof. In order to estimate the L^q norm of the operator S_δ we first estimate S_δ point-wise, and so

$$|(S_\delta f)(x)| \leq \int_{\mathbf{R}^n} \chi_{E_\delta}(x, y) |f(y)| dy. \quad (2.3.2)$$

The strategy is simply to transform the positive operator on the righthand side of the inequality in (2.3.2), into one which involves an oscillatory integral operator of the second kind, so that we can apply the hypotheses of the theorem. We fix a non-negative cut-off function $\varphi \in C_0^\infty(\mathbf{R})$ such that $\varphi(t) = 1$ whenever $|t| < 1$, so that point-wise we have the inequality

$$\chi_{E_\delta}(x, y) \leq \varphi\left(\frac{\Phi(x, y)}{\delta}\right).$$

Then, since $\psi \equiv 1$ on $U \times V$, we can write

$$\begin{aligned} |(S_\delta f)(x)| &\leq \int_{\mathbf{R}^n} \chi_{E_\delta}(x, y) |f(y)| dy \\ &\leq \int_{\mathbf{R}^n} \psi(x, y) \varphi\left(\frac{\Phi(x, y)}{\delta}\right) |f(y)| dy \\ &= \int_{-\infty}^{\infty} \widehat{\varphi}(\lambda) (T_{\frac{2\pi\lambda}{\delta}} |f|)(x) d\lambda \end{aligned}$$

where $(T_{\frac{2\pi\lambda}{\delta}} |f|)(x) = \int_{\mathbf{R}^n} e^{\frac{2\pi i \lambda}{\delta} \Phi(x, y)} \psi(x, y) |f(y)| dy$.

Hence, applying the Minkowski integral inequality and the hypotheses of the theorem we obtain

$$\begin{aligned}
\|S_\delta(f)\|_{L^q} &\leq \int_{-\infty}^{\infty} \|T_{\frac{2\pi\lambda}{\delta}}|f|\|_{L^q} |\widehat{\varphi}(\lambda)| d\lambda \\
&\leq C \int_{-\infty}^{\infty} |\widehat{\varphi}(\lambda)| \min(1, (\delta/2\pi|\lambda|)^\eta) d\lambda \cdot \|f\|_{L^p}.
\end{aligned}$$

The constant 2π plays no crucial role in the remaining analysis and so we may drop it. We write

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{\varphi}(\lambda)| \min(1, (\delta/|\lambda|)^\eta) d\lambda &= \left(\int_{\{\lambda: |\lambda| > \delta\}} + \int_{\{\lambda: |\lambda| \leq \delta\}} \right) |\widehat{\varphi}(\lambda)| \min(1, (\delta/|\lambda|)^\eta) d\lambda \\
&= I + II
\end{aligned}$$

We see that $II \lesssim \delta$. Now $\widehat{\varphi} \in \mathcal{S}(\mathbf{R})$ implies $|\widehat{\varphi}(\lambda)| \lesssim \min(1, |\lambda|^{-1})$, hence we split I further and write

$$\int_{\{\lambda: |\lambda| > \delta\}} = \int_{\{\lambda: \delta < |\lambda| \leq 1\}} + \int_{\{\lambda: |\lambda| > 1\}}.$$

Considering separately the cases $0 \leq \eta < 1$, $\eta = 1$, and $\eta > 1$, we obtain

$$I \lesssim \begin{cases} \delta^\eta & 0 \leq \eta < 1, \\ \delta \log\left(\frac{1}{\delta}\right) & \eta = 1, \\ \delta & \eta > 1. \end{cases}$$

Hence, combining I and II we conclude that

$$\|S_\delta\|_{L^p \rightarrow L^q} \leq AC \begin{cases} \delta^\eta & 0 \leq \eta < 1, \\ \delta \log\left(\frac{1}{\delta}\right) & \eta = 1, \\ \delta & \eta > 1. \end{cases}$$

□

We will now apply Theorem 2.3.1 to improve Proposition 2.2.1. The scheme of our approach is essentially a repetition of Theorem 2.3.1 with a few minor adjustments, and so for this reason we will keep the details sparse. However, before we present the proof we will require a useful proposition which will validate our use of the Fourier inversion formula during the latter stage of the proof.

Proposition 2.3.2. *Let $\varphi(x) = |\psi(x)| \forall x \in \mathbf{R}$. If $\psi \in \mathcal{S}(\mathbf{R})$, then the following statements hold:*

(a) *For every non-negative $g \in \mathcal{S}(\mathbf{R})$ such that $\widehat{g}(0) = 1$ we have*

$$\varphi(s) = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \widehat{g}(\epsilon y) \widehat{\varphi}(y) e^{2\pi i y s} dy.$$

(b) *We have the point-wise estimate*

$$|\widehat{\varphi}(x)| \leq C|x|^{-1}.$$

Proof. Proof of part (a).

Fix $\epsilon > 0$ to be small. By making a change of variables $u = \epsilon y$, and changing the order of integration, we have for each s that

$$\begin{aligned} \int_{-\infty}^{\infty} \widehat{g}(\epsilon y) \widehat{\varphi}(y) e^{2\pi i y s} dy &= \int_{-\infty}^{\infty} \widehat{g}(\epsilon y) \left(\int_{-\infty}^{\infty} e^{-2\pi i y t} \varphi(t) dt \right) e^{2\pi i y s} dy \\ &= \frac{1}{\epsilon} \int_{-\infty}^{\infty} \widehat{g}(y) \left(\int_{-\infty}^{\infty} e^{\frac{-2\pi i y t}{\epsilon}} \varphi(t) dt \right) e^{\frac{2\pi i y s}{\epsilon}} dy \\ &= \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{2\pi i \left(\frac{s-t}{\epsilon}\right) y} \widehat{g}(y) dy \right) \varphi(t) dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\epsilon} g\left(\frac{s-t}{\epsilon}\right) \varphi(t) dt \\ &= (g_{\epsilon} * \varphi)(s) \end{aligned}$$

where $g_\epsilon(y) = \frac{1}{\epsilon}g(\frac{y}{\epsilon})$.

The change of order of integration is valid since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(t)| |\widehat{g}(y)| dy dt < \infty$, and as $g \in \mathcal{S}(\mathbf{R})$ by hypothesis, the Fourier inversion formula holds for the function g .

We next note that by making a changing variables $u = \epsilon^{-1}y$, and using the fact that $\int_{-\infty}^{\infty} g_\epsilon(y) dy = \widehat{g}(0) = 1$, we can write

$$\begin{aligned} (g_\epsilon * \varphi)(s) - \varphi(s) &= \int_{-\infty}^{\infty} g_\epsilon(y)(\varphi(s - y) - \varphi(s)) dy \\ &= \int_{-\infty}^{\infty} g(y)(\varphi(s - \epsilon y) - \varphi(s)) dy. \end{aligned}$$

We will now want to take the limit as $\epsilon \rightarrow 0$. Since $\psi \in \mathcal{S}(\mathbf{R})$, we have that φ is bounded. Hence, for each fixed s , the resulting integrand $g(y)(\varphi(s - \epsilon y) - \varphi(s))$ is uniformly bounded in absolute value by $2\|\varphi\|_{L^\infty}|g(y)|$, an integrable function in y . So by the Lebesgue dominated convergence theorem we can take the limit on the inside of the integral. Since φ is continuous at every point, it follows that, for each fixed s , the difference $|\varphi(s - \epsilon y) - \varphi(s)|$ tends to zero as $\epsilon \rightarrow 0$, and so the proof of part (a) is complete.

Proof of part (b).

We first observe that φ has bounded variation on \mathbf{R} . That is, we want to prove that there exists a finite constant A , such that for any sequence of points satisfying $-\infty < x_0 < x_1 < \dots < x_n < \infty$, the sum below satisfies

$$\sum_{1 \leq j \leq n} |\varphi(x_j) - \varphi(x_{j-1})| \leq A. \quad (2.3.3)$$

This is seen as follows. We have for any u, v , with $u > v$, that

$$\begin{aligned}
|\varphi(u) - \varphi(v)| &= ||\psi(u)| - |\psi(v)|| \\
&\leq |\psi(u) - \psi(v)| \\
&= \left| \int_v^u \psi'(s) ds \right| \\
&\leq \int_v^u |\psi'(s)| ds.
\end{aligned}$$

Hence, the sum in (2.3.3) is bounded above by the integral $\int_{x_0}^{x_n} |\psi'(s)| ds$, which is in turn bounded above by the integral $\int_{-\infty}^{\infty} |\psi'(s)| ds$. The latter integral is finite since $\psi \in \mathcal{S}(\mathbf{R})$, and so, we may take the A in (2.3.3) to be equal to $\int_{-\infty}^{\infty} |\psi'(s)| ds$.

Therefore, since φ is continuous, of bounded variation, and $\lim_{|x| \rightarrow \infty} \psi(x) = 0$, we can apply integration by parts once to obtain

$$\int_{-\infty}^{\infty} e^{-2\pi i x s} \varphi(s) ds = \left(\frac{-1}{2\pi i x} \right) \int_{-\infty}^{\infty} \varphi(s) (e^{-2\pi i x s})' ds \quad (2.3.4)$$

$$= \left(\frac{-1}{2\pi i x} \right) [\varphi(s) e^{-2\pi i x s}]_{-\infty}^{\infty} + \frac{1}{2\pi i x} \int_{-\infty}^{\infty} e^{2\pi i x s} d\varphi(s) \quad (2.3.5)$$

$$= \frac{1}{2\pi i x} \int_{-\infty}^{\infty} e^{2\pi i x s} d\varphi(s). \quad (2.3.6)$$

The integral in (2.3.6) is at most, in absolute value, the integral $\int_{-\infty}^{\infty} |d\varphi(s)|$, which is the total variation of φ , and this has just been proved earlier to be at most $\int_{-\infty}^{\infty} |\psi'(s)| ds = A < \infty$, and so the proof of part (b) is complete. □

We can now go ahead and improve Proposition 2.2.1.

Theorem 2.3.3. *For $\psi \in \mathcal{S}(\mathbf{R})$, and $a \in L_c^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, the integral operator defined by*

$$(\mathcal{L}_\lambda f)(x) = \int_{\mathbf{R}^n} a(x, y) \psi(\lambda \Phi(x, y)) f(y) dy \quad (2.3.7)$$

with $\Phi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, and $\det(\frac{\partial^2 \Phi(x,y)}{\partial x_i \partial y_j}) \neq 0$ on the support of a , obeys the following $L^2 \rightarrow L^2$ estimate:

$$\|\mathcal{L}_\lambda\|_{L^2 \rightarrow L^2} \lesssim \begin{cases} \lambda^{-1/2} & n = 1, \\ \lambda^{-1} \log \lambda & n = 2, \\ \lambda^{-1} & n \geq 3. \end{cases}$$

Proof. The strategy is to simply put ourselves in a position where we can apply the L^2 estimate of Hörmander, as given in Proposition 1.8.3, in exactly the same manner as we applied the oscillatory integral operator norm estimate in Theorem 2.3.1. We start off by estimating the operator \mathcal{L}_λ pointwise, and so

$$\begin{aligned} |(\mathcal{L}_\lambda f)(x)| &\leq \int_{\mathbf{R}^n} |a(x, y)| |\psi(\lambda \Phi(x, y))| |f(y)| dy \\ &\leq \|a\|_{L^\infty} \int_{\mathbf{R}^n} \chi_{\text{supp } a}(x, y) |\psi(\lambda \Phi(x, y))| |f(y)| dy. \end{aligned}$$

Now, for the sake of notational convenience, let us set for each $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$

$$\Gamma_\Phi(x, y) = \det\left(\frac{\partial^2 \Phi(x, y)}{\partial x_i \partial y_j}\right).$$

We note that since $\Phi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, and that the determinant function is continuous, it follows that for every such Φ that the function Γ_Φ is continuous. Consider the set

$$\Omega = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : \Gamma_\Phi(x, y) \neq 0\}.$$

By hypothesis, we have that $\Gamma_\Phi \neq 0$ on $\text{supp } a$, and so it follows trivially that $\text{supp } a \subset \Omega$. Moreover, since we have the identity $\Omega = \Gamma_\Phi^{-1}(\mathbf{R} \setminus \{0\})$, it then follows, by the fact that $\mathbf{R} \setminus \{0\}$ is an open set and that Γ_Φ is continuous, that Ω is also an open set.

Hence, since $\text{supp } a$ is a compact set, and $\text{supp } a \subset \Omega$ with Ω being an open set, there then exists a nonnegative $\Psi \in C_c^\infty(\Omega)$ such that $\text{supp } \Psi \subset \Omega$ with $\Psi \equiv 1$ on $\text{supp } a$. And so, we have a function Ψ where point-wise we have the inequality

$$\chi_{\text{supp } a}(x, y) \leq \Psi(x, y)$$

with $\det(\frac{\partial^2 \Phi(x, y)}{\partial x_i \partial y_j}) \neq 0$ on $\text{supp } \Psi$. This will now allow us to engineer an oscillatory integral operator of the second kind that has a smooth “cut-off” function Ψ which satisfies the conditions of Proposition 1.8.3, thus allowing us to utilise it in our calculation.

Hence we can write

$$|(\mathcal{L}_\lambda f)(x)| \lesssim \int_{\mathbf{R}^n} \Psi(x, y) \varphi(\lambda \Phi(x, y)) |f(y)| dy, \quad (2.3.8)$$

where we have set $\varphi(s) = |\psi(s)|$.

Fixing now a non-negative $g \in \mathcal{S}(\mathbf{R})$ such that $\widehat{g}(0) = 1$, we can now use the first part of Proposition 2.3.2 to write the integral operator on the right of the inequality in (2.3.8) as

$$\int_{\mathbf{R}^n} \Psi(x, y) \left(\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \widehat{g}(\epsilon z) \widehat{\varphi}(z) e^{2\pi i z \lambda \Phi(x, y)} dz \right) |f(y)| dy,$$

and this is of course equal to

$$\int_{\mathbf{R}^n} \lim_{\epsilon \rightarrow 0} \Psi(x, y) \int_{-\infty}^{\infty} \widehat{g}(\epsilon z) \widehat{\varphi}(z) e^{2\pi i z \lambda \Phi(x, y)} dz |f(y)| dy. \quad (2.3.9)$$

We now fix $\epsilon > 0$ to be small, and for fixed $x \in \mathbf{R}^n$ we set

$$h_{\epsilon,x}(y) = \Psi(x, y) \int_{-\infty}^{\infty} \widehat{g}(\epsilon z) \widehat{\varphi}(z) e^{2\pi i z \lambda \Phi(x, y)} dz |f(y)|.$$

For each fixed $\epsilon > 0$ and for each fixed $x \in \mathbf{R}^n$ we note that

$$h_{\epsilon,x}(y) = \Psi(x, y) (g_{\epsilon} * \varphi)(\lambda \Phi(x, y)) |f(y)|,$$

and so, since $|(g_{\epsilon} * \varphi)(\lambda \Phi(x, y))| \leq \|\varphi\|_{L^{\infty}}$, we observe that

$$|h_{\epsilon,x}(y)| \leq \|\varphi\|_{L^{\infty}} |\Psi(x, y)| |f(y)| = d_x(y)$$

for each $y \in \mathbf{R}^n$. It is easy to see that for each fixed $x \in \mathbf{R}^n$ that $d_x \in L^1(\mathbf{R}^n)$, and hence we can apply the Lebesgue dominated convergence theorem to write (2.3.9) as

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbf{R}^n} h_{\epsilon,x}(y) dy.$$

Now, for each fixed $\epsilon > 0$, and using the substitution $s = \epsilon z$, we can write

$$\begin{aligned} \int_{\mathbf{R}^n} h_{\epsilon,x}(y) dy &= \int_{-\infty}^{\infty} \left(\int_{\mathbf{R}^n} e^{2\pi i z \lambda \Phi(x, y)} \Psi(x, y) |f(y)| dy \right) \widehat{g}(\epsilon z) \widehat{\varphi}(z) dz \\ &= \int_{-\infty}^{\infty} \epsilon^{-1} \left(\int_{\mathbf{R}^n} e^{2\pi i z \frac{\lambda}{\epsilon} \Phi(x, y)} \Psi(x, y) |f(y)| dy \right) \widehat{g}(z) \widehat{\varphi}(z/\epsilon) dz \\ &= \int_{-\infty}^{\infty} \epsilon^{-1} (T_{2\pi z \frac{\lambda}{\epsilon}} |f|)(x) \widehat{g}(z) \widehat{\varphi}(z/\epsilon) dz. \end{aligned}$$

where $(T_{2\pi z \frac{\lambda}{\epsilon}} |f|)(x) = \int_{\mathbf{R}^n} e^{2\pi i z \frac{\lambda}{\epsilon} \Phi(x, y)} \Psi(x, y) |f(y)| dy$.

Hence we have so far that

$$|(\mathcal{L}_{\lambda} f)(x)| \lesssim \lim_{\epsilon \rightarrow 0} \left| \int_{-\infty}^{\infty} \epsilon^{-1} (T_{2\pi z \frac{\lambda}{\epsilon}} |f|)(x) \widehat{g}(z) \widehat{\varphi}(z/\epsilon) dz \right|.$$

We then have by an application of Fatou's lemma that

$$\int_{\mathbf{R}^n} |(\mathcal{L}_\lambda f)(x)|^2 dx \lesssim \liminf_{\epsilon \rightarrow 0} \int_{\mathbf{R}^n} \left| \int_{-\infty}^{\infty} \epsilon^{-1} (T_{2\pi z \frac{\lambda}{\epsilon}} |f|)(x) \widehat{g}(z) \widehat{\varphi}(z/\epsilon) dz \right|^2 dx,$$

and hence it follows that

$$\|\mathcal{L}_\lambda(f)\|_{L^2(\mathbf{R}^n)} \lesssim \liminf_{\epsilon \rightarrow 0} \left\| \int_{-\infty}^{\infty} \epsilon^{-1} T_{2\pi z \frac{\lambda}{\epsilon}} (|f|) \widehat{g}(z) \widehat{\varphi}(z/\epsilon) dz \right\|_{L^2(\mathbf{R}^n)}.$$

Applying the Minkowski integral inequality, and the L^2 norm estimate of Hörmander in Proposition 1.8.3 for the operator $T_{2\pi z \frac{\lambda}{\epsilon}}$, we obtain

$$\begin{aligned} \|\mathcal{L}_\lambda(f)\|_{L^2(\mathbf{R}^n)} &\lesssim \liminf_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \epsilon^{-1} \|T_{2\pi z \frac{\lambda}{\epsilon}} |f|\|_{L^2(\mathbf{R}^n)} |\widehat{g}(z)| |\widehat{\varphi}(z/\epsilon)| dz \\ &\lesssim \liminf_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \epsilon^{-1} \min(1, (\epsilon/2\pi\lambda|z|)^{\frac{n}{2}}) |\widehat{g}(z)| |\widehat{\varphi}(z/\epsilon)| dz \cdot \|f\|_{L^2(\mathbf{R}^n)}. \end{aligned}$$

The constant 2π plays no crucial role in the remaining analysis and so we may drop it.

Since $\epsilon > 0$ is small, we may take it to be $0 < \epsilon < 1$. We also have by part (b) of Proposition 2.3.2 that $\widehat{\varphi}$ satisfies the bound $|\widehat{\varphi}(z)| \leq C|z|^{-1}$, and so we decompose the region of integration as follows

$$\mathbf{R} = \left\{ z : |z| < \frac{\epsilon}{\lambda} \right\} \cup \left\{ z : \frac{\epsilon}{\lambda} \leq |z| \leq \epsilon \right\} \cup \{ z : \epsilon < |z| \leq 1 \} \cup \{ z : 1 < |z| \}.$$

We will first estimate the integral over the region $\{z : |z| < \frac{\epsilon}{\lambda}\}$, and then we will deal with the other three regions afterwards. For the sake of notational convenience we will denote the remaining three regions as R_1 , R_2 , and R_3 respectively.

We see that

$$\begin{aligned} \int_{\{z: |z| < \frac{\epsilon}{\lambda}\}} \epsilon^{-1} \min \left(1, \left(\frac{\epsilon}{\lambda|z|} \right)^{\frac{n}{2}} \right) |\widehat{g}(z)| |\widehat{\varphi}(z/\epsilon)| dz &\lesssim \frac{1}{\epsilon} \|\widehat{g}\|_{L^\infty} \|\widehat{\varphi}\|_{L^\infty} \int_{\{z: |z| < \frac{\epsilon}{\lambda}\}} dz. \\ &\lesssim \frac{1}{\lambda}. \end{aligned}$$

We then analyse the integral on the remaining regions R_1 , R_2 , and R_3 by considering the separate cases when $n = 1$, $n = 2$, and $n \geq 3$. Hence, we observe that

$$\int_{R_1 \cup R_2 \cup R_3} \epsilon^{-1} \min \left(1, \left(\frac{\epsilon}{\lambda|z|} \right)^{\frac{n}{2}} \right) |\widehat{g}(z)| |\widehat{\varphi}(z/\epsilon)| dz \lesssim \begin{cases} \lambda^{-1/2} + O(\epsilon^{1/2} \lambda^{-1/2}) & n = 1, \\ \lambda^{-1} \log \lambda + O(\epsilon \lambda^{-1}) & n = 2, \\ \lambda^{-1} + O(\epsilon^{n/2} \lambda^{-n/2}) & n \geq 3. \end{cases}$$

Therefore, putting all of the estimates for the separate regions together, we then have that

$$\frac{\|\mathcal{L}_\lambda(f)\|_{L^2(\mathbf{R}^n)}}{\|f\|_{L^2(\mathbf{R}^n)}} \lesssim \liminf_{\epsilon \rightarrow 0} \begin{cases} \lambda^{-1/2} + O(\epsilon^{1/2} \lambda^{-1/2}) & n = 1, \\ \lambda^{-1} \log \lambda + O(\epsilon \lambda^{-1}) & n = 2, \\ \lambda^{-1} + O(\epsilon^{n/2} \lambda^{-n/2}) & n \geq 3. \end{cases}$$

Hence, taking the limit, we finally conclude that

$$\|\mathcal{L}_\lambda\|_{L^2 \rightarrow L^2} \lesssim \begin{cases} \lambda^{-1/2} & n = 1, \\ \lambda^{-1} \log \lambda & n = 2, \\ \lambda^{-1} & n \geq 3. \end{cases}$$

□

2.3.1 Sharpness of the result in Theorem 2.3.3

Let Q denote the closed unit interval $[0, 1]$ and E_δ denote the sub-level set $\{(x, y) \in Q^n \times Q^n : |\Phi(x, y)| < \delta\}$. In this section we will provide examples of Φ that indicate the sharpness of the result obtained in Theorem 2.3.3. For this purpose, we will consider the sub-level set operator

$$(\mathcal{L}_{\delta^{-1}}f)(x) = \int_{\mathbf{R}^n} a(x, y) \psi\left(\frac{\Phi(x, y)}{\delta}\right) f(y) dy$$

where $a(x, y) = \chi_{Q^n \times Q^n}(x, y)$ and $\psi \in C_0^\infty(\mathbf{R})$ is a fixed non-negative cut-off function such that $\psi(t) = 1$ whenever $|t| < 1$, so that point-wise we have the inequality

$$\chi_{E_\delta}(x, y) \leq \psi\left(\frac{\Phi(x, y)}{\delta}\right).$$

We will require the the following lemma in our calculations.

Lemma 2.3.4. *For any p and q and with $\mathcal{L}_{\delta^{-1}}$ being the particular sub-level set operator defined as above, the following inequality holds $|E_\delta| \leq \|\mathcal{L}_{\delta^{-1}}\|_{L^p \rightarrow L^q}$.*

Proof. The proof is very straightforward and simply follows from an application of Fubini's theorem and Hölder's inequality. We proceed as follows

$$\begin{aligned} |E_\delta| &= \int_{Q^n} \int_{Q^n} \chi_{E_\delta}(x, y) dx dy \\ &= \int_{Q^n} \left(\int_{Q^n} \chi_{E_\delta}(x, y) dy \right) dx \\ &\leq \int_{\mathbf{R}^n} \chi_{Q^n}(x) \left(\int_{\mathbf{R}^n} a(x, y) \psi\left(\frac{\Phi(x, y)}{\delta}\right) \chi_{Q^n}(y) dy \right) dx \\ &\leq \left(\int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} a(x, y) \psi\left(\frac{\Phi(x, y)}{\delta}\right) \chi_{Q^n}(y) dy \right|^q dx \right)^{1/q} \\ &= \|\mathcal{L}_{\delta^{-1}} \chi_{Q^n}\|_{L^q(\mathbf{R}^n)} \leq \|\mathcal{L}_{\delta^{-1}}\|_{L^p \rightarrow L^q}, \end{aligned}$$

and thus the proof of the lemma is complete. □

From the lemma above, we see that in order to demonstrate sharpness for each

dimension in question, namely $n = 1$, $n = 2$, and $n \geq 3$, it will suffice to produce examples of Φ for which $|E_\delta|$ will have the appropriate measure. We note that one can easily check for all the examples that follow that one indeed has $\det(\frac{\partial^2 \Phi(x,y)}{\partial x_i \partial y_j}) \neq 0$ on the support of a , where $a(x, y) = \chi_{Q^n \times Q^n}(x, y)$. Moreover, since the sub-level set operators \mathcal{L}_λ in the previous section are considered for large λ i.e. $\lambda > 1$, we remind the reader that we are also doing the same here with the operator $\mathcal{L}_{\delta^{-1}}$, and so the reader should have in mind that δ^{-1} is large i.e. $\delta < 1$.

Example 2.3.5. For $n = 1$, by considering $\Phi(x, y) = |x - y|^2$, we obtain $|E_\delta| \sim \delta^{1/2}$. We can see this as follows.

Via Fubini's theorem, we write

$$|E_\delta| = \int_Q |\{y \in Q : |x - y| < \delta^{1/2}\}| dx.$$

Since $\delta < 1$, we obtain that $|\{y \in Q : |x - y| < \delta^{1/2}\}| \sim \delta^{1/2}$, and so we have that

$$|E_\delta| \sim \int_Q \delta^{1/2} dx \sim \delta^{1/2}.$$

Example 2.3.6. For $n = 2$, by considering $\Phi(x_1, x_2, y_1, y_2) = |x_1 - y_1|^2 - |x_2 - y_2|^2$, we obtain $|E_\delta| \sim \delta \log(\frac{1}{\delta})$. We can see this as follows.

Via Fubini's theorem, we write

$$|E_\delta| = \int_0^1 \int_0^1 |\{(x_1, x_2) \in Q \times Q : ||x_1 - y_1|^2 - |x_2 - y_2|^2| < \delta\}| dy_1 dy_2.$$

Since

$$\begin{aligned} & |\{(x_1, x_2) \in Q \times Q : ||x_1 - y_1|^2 - |x_2 - y_2|^2| < \delta\}| \\ & \sim |\{(x_1, x_2) \in Q \times Q : ||x_1|^2 - |x_2|^2| < \delta\}|, \end{aligned}$$

we now obtain that

$$|E_\delta| \sim \int_0^1 \int_0^1 \chi_{\{(z_1, z_2): |z_1|^2 - |z_2|^2 < \delta\}}(x_1, x_2) dx_1 dx_2.$$

We apply the transformations $x_1 = u - s$, $x_2 = u + s$ and hence the above double integral becomes

$$2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 \chi_{\{(z_1, z_2): |z_1|^2 - |z_2|^2 < \delta\}}(u - s, u + s) du ds, \quad (2.3.10)$$

where of course, the factor 2 arises from the Jacobian of the transformation.

We check that the integrand in (2.3.10) is actually equal to the function $\chi_{\{(r, t): |rt| < \frac{\delta}{4}\}}(u, s)$, and thus we see that the integral in (2.3.10) is actually equal to

$$2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 \chi_{\{(r, t): |rt| < \frac{\delta}{4}\}}(u, s) du ds.$$

Hence, applying Fubini's theorem again, we see that

$$\begin{aligned} |E_\delta| &\sim 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 \chi_{\{(r, t): |rt| < \frac{\delta}{4}\}}(u, s) du ds \\ &= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_0^1 \chi_{\{(r, t): |rt| < \frac{\delta}{4}\}}(u, s) du \right) ds \\ &= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \left\{ u \in Q : |u| < \frac{\delta}{4|s|} \right\} \right| ds \\ &= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \min \left(1, \frac{\delta}{2|s|} \right) ds. \end{aligned}$$

Thus, we finally obtain that

$$\begin{aligned}
|E_\delta| &\sim \int_{\{s: |s| < \frac{\delta}{2}\}} ds + \int_{\{s: \frac{\delta}{2} \leq |s| \leq \frac{1}{2}\}} \frac{\delta}{|s|} ds \\
&\sim \delta + \delta \log \left(\frac{1}{\delta} \right) \\
&\sim \delta \log \left(\frac{1}{\delta} \right).
\end{aligned}$$

Example 2.3.7. For $n \geq 3$, by considering $\Phi(x, y) = x \cdot y$, we obtain $|E_\delta| \gtrsim \delta$.

We can see this as follows. Via Fubini's theorem, we write

$$|E_\delta| = \int_{Q^n} |\{x \in Q^n : |x \cdot (y/|y|)| < \delta/|y|\}| dy.$$

Observe that $|\{x \in Q^n : |x \cdot \frac{y}{|y|}| < \frac{\delta}{|y|}\}| \sim \min(1, \frac{\delta}{|y|})$. Hence, we can write

$$\begin{aligned}
|E_\delta| &\sim \int_{\{y: |y| < \delta\}} dy + \int_{\{y: \delta \leq |y|\}} \frac{\delta}{|y|} dy \\
|E_\delta| &\gtrsim \int_{\{y: |y| < \delta\}} dy + \int_{\{y: \delta \leq |y| \leq 1\}} \frac{\delta}{|y|} dy \\
&= \delta^n \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} + \delta \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_\delta^1 r^{n-2} dr \\
&= \delta^n \frac{2\pi^{n/2}}{n\Gamma(n/2)} + \delta \frac{2\pi^{n/2}}{\Gamma(n/2)} \left(1 - \frac{\delta^{n-1}}{n-1} \right) \\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)} \left(\delta - \frac{\delta^n}{n(n-1)} \right),
\end{aligned}$$

where Γ is the Gamma function defined by $\Gamma(\sigma) = \int_0^\infty x^{\sigma-1} e^{-x} dx$, with the integral being convergent for all $\sigma > 0$.

Since $\delta < 1$ and $n \geq 3$, it follows that

$$\frac{5}{6}\delta < \delta - \frac{\delta^n}{n(n-1)} < \delta,$$

and thus, we have finally obtain that

$$|E_\delta| \gtrsim C_n \delta$$

where $C_n = \frac{\pi^{n/2}}{\Gamma(n/2)}$.

2.4 Further sub-level set results

2.4.1 A local sub-level set result

We will now obtain a local sub-level set estimate by utilising Morse's lemma. The proof of Morse's lemma can be found in [29]. Let us begin with Φ real valued and in $C^\infty(\mathbf{R}^n)$, with $\Phi(0) = \nabla\Phi(0) = 0$ and $\det(\frac{\partial^2\Phi(0)}{\partial x_i\partial x_j}) \neq 0$. Let U_0 be a small neighbourhood about the origin, and let p denote the number of $+1$ s in the diagonal of the diagonalised matrix of $\frac{\partial^2\Phi(0)}{\partial x_i\partial x_j}$ and let q be the number of -1 s in the diagonal of the diagonalised matrix of $\frac{\partial^2\Phi(0)}{\partial x_i\partial x_j}$. Set $E_\delta = \{x \in U_0 : |\Phi(x)| < \delta\}$. Then under the above assumptions on Φ we have the following local result.

Theorem 2.4.1.

$$|E_\delta| \lesssim \begin{cases} \delta^{n/2} & p = n \quad q = 0 \quad \text{or} \quad p = 0 \quad q = n, \\ \delta & p, q \neq 0, \max(p, q) > 1, \\ \delta \log\left(\frac{1}{\delta}\right) & p, q \neq 0, \max(p, q) = 1. \end{cases}$$

Proof. We appeal to the change of variables guaranteed by Morse's lemma: Since $\Phi(0) = \nabla\Phi(0) = 0$ and $\det(\frac{\partial^2\Phi(0)}{\partial x_i\partial x_j}) \neq 0$, there exists a diffeomorphism ψ from U_0 in the x space, to a neighbourhood V_0 of the origin in the y space under which Φ

is transformed into $\tilde{\Phi}$ where

$$\tilde{\Phi}(y) = \begin{cases} \pm(y_1^2 + \dots + y_n^2), \\ y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_{p+q}^2, \end{cases}$$

depending on the number p of $+1$ s and the number q of -1 s in the diagonal of the diagonalised matrix of $\frac{\partial^2 \Phi(0)}{\partial x_i \partial x_j}$. In the second case, since both $p, q \neq 0$, we make the distinction as to whether $\max(p, q) > 1$ or $\max(p, q) = 1$. Since ψ is a diffeomorphism, the measure of our set E_δ is preserved up to a constant factor that is dependent on the bounds for the determinant of the Jacobian of the diffeomorphism. So in the first case we have

$$\begin{aligned} |E_\delta| &\sim |\{y \in V_0 : |y_1^2 + \dots + y_n^2| < \delta\}| \\ &\leq |\{y \in Q^n : |y| < \delta^{1/2}\}| \\ &\lesssim \delta^{n/2}. \end{aligned}$$

Setting

$$A_\delta = \{(x, y) \in Q^p \times Q^q : (|x|^2 - \delta)^{1/2} < |y| < (|x|^2 + \delta)^{1/2}\},$$

we have in the second case

$$|E_\delta| \sim |\{y \in V_0 : |y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_{p+q}^2| < \delta\}| \quad (2.4.1)$$

$$\leq |\{y \in Q^{p+q} : |y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_{p+q}^2| < \delta\}| \quad (2.4.2)$$

$$= |\{(x, y) \in Q^p \times Q^q : ||x|^2 - |y|^2| < \delta\}| \quad (2.4.3)$$

$$= |\{(x, y) \in Q^p \times Q^q : (|x|^2 - \delta)^{1/2} < |y| < (|x|^2 + \delta)^{1/2}\}| \quad (2.4.4)$$

$$= \int_{Q^p} \int_{Q^q} \chi_{A_\delta}(y, x) dy dx. \quad (2.4.5)$$

For each fixed $x \in Q^p$, if we vary $y \in Q^q$ in the set A_δ , we obtain the section of

A_δ along this particular x . The section of A_δ along x is denoted as A_δ^x , and is equal to the set $\{y \in Q^q : (x, y) \in A_\delta\}$. Fubini's theorem tells us that

$$\int_{Q^p} \left(\int_{Q^q} \chi_{A_\delta}(y, x) dy \right) dx = \int_{Q^p} |A_\delta^x| dx = |A_\delta|.$$

We proceed by splitting the x region of integration into two parts as follows:

$$Q^p = \{x \in Q^p : |x| \lesssim \sqrt{\delta}\} \cup \{x \in Q^p : \sqrt{\delta} < |x|\}.$$

Then

$$|A_\delta| = \int_{|x| \lesssim \sqrt{\delta}} |A_\delta^x| dx + \int_{\sqrt{\delta} < |x| \lesssim 1} |A_\delta^x| dx = T_1 + T_2.$$

It will now be our wish to estimate T_1 and T_2 depending on whether $\max(p, q) > 1$ or $\max(p, q) = 1$.

Case (1) $\max(p, q) > 1$.

(i) Estimate for T_1 :

For fixed x such that $|x| \lesssim \sqrt{\delta}$ we have that

$$\begin{aligned} A_\delta^x &= \{y \in Q^q : -\delta \lesssim |y|^2 \lesssim \delta\} \\ &= \{y \in Q^q : -\delta \lesssim |y|^2\} \cap \{y \in Q^q : |y|^2 \lesssim \delta\} \\ &\subset \mathbf{R}^q \cap \{y \in Q^q : |y|^2 \lesssim \delta\} \\ &= \{y \in Q^q : |y|^2 \lesssim \delta\}. \end{aligned}$$

Hence $|A_\delta^x| \lesssim \delta^{q/2}$, which in turn implies $T_1 \lesssim \delta^{(p+q)/2} = \delta^{n/2}$.

(ii) Estimate for T_2 :

For fixed x we always have that

$$A_\delta^x = \{y \in Q^q : |x| \left(1 - \frac{\delta}{|x|^2}\right)^{1/2} < |y| < |x| \left(1 + \frac{\delta}{|x|^2}\right)^{1/2}\}.$$

Now, for fixed x such that $\sqrt{\delta} < |x|$ and $\frac{\delta}{|x|^2}$ being very small, we have the binomial approximation

$$\left(1 \pm \frac{\delta}{|x|^2}\right)^{1/2} \sim 1 \pm \frac{\delta}{2|x|^2}.$$

Therefore A_δ^x is now an annulus with radial boundaries $r_1 = |x| \left(1 - \frac{\delta}{2|x|^2}\right)$ and $r_2 = |x| \left(1 + \frac{\delta}{2|x|^2}\right)$. Hence

$$|A_\delta^x| \sim r_2^q - r_1^q = |x|^q \left(1 + \frac{\delta}{2|x|^2}\right)^q - |x|^q \left(1 - \frac{\delta}{2|x|^2}\right)^q \sim \frac{q}{2} |x|^{q-2} \delta \sim |x|^{q-2} \delta.$$

Therefore

$$T_2 \sim \delta \int_{\sqrt{\delta} < |x| \lesssim 1} |x|^{q-2} dx = \delta \int_{\sqrt{\delta} < |x| \lesssim 1} |x|^{q-2} dx \sim \delta \int_{\sqrt{\delta}}^1 r^{q-2} r^{p-1} dr = \delta \int_{\sqrt{\delta}}^1 r^{p+q-3} dr.$$

Now $\max(p, q) > 1$ implies that $p \geq 2$ or $q \geq 2$, therefore $p + q - 3 \geq 0$. Thus

$$\int_{\sqrt{\delta}}^1 r^{p+q-3} dr \sim 1, \text{ which implies } T_2 \sim \delta.$$

So we have

$$T_1 \lesssim \delta^{n/2} \text{ and } T_2 \sim \delta.$$

Case (2) $\max(p, q) = 1$.

The above condition on p and q implies that $p = q = 1$, and so

$$A_\delta = \{(x, y) \in Q \times Q : (|x|^2 - \delta)^{1/2} < |y| < (|x|^2 + \delta)^{1/2}\}.$$

(i) Estimate for T_1 :

We conclude in exactly the same manner as before, except now with $p = q = 1$ that $|A_\delta^x| \lesssim \delta^{1/2}$, which in turn implies $T_1 \lesssim \delta$.

(ii) Estimate for T_2 :

Again using the previous calculation, we have with $p = q = 1$ that

$$T_2 \sim \delta \int_{\sqrt{\delta}}^1 r^{-1} dr \sim \delta \log \left(\frac{1}{\delta} \right).$$

So we have

$$T_1 \lesssim \delta \quad \text{and} \quad T_2 \sim \delta \log \left(\frac{1}{\delta} \right).$$

Hence in case (1) we have

$$|E_\delta| \leq |A_\delta| = T_1 + T_2 \lesssim \delta^{n/2} + \delta \lesssim \delta,$$

and in case (2) we have

$$|E_\delta| \leq |A_\delta| = T_1 + T_2 \lesssim \delta + \delta \log \left(\frac{1}{\delta} \right) \lesssim \delta \log \left(\frac{1}{\delta} \right).$$

Collecting the two cases together we have finally that

$$|E_\delta| \lesssim \begin{cases} \delta & \max(p, q) > 1, \\ \delta \log\left(\frac{1}{\delta}\right) & \max(p, q) = 1. \end{cases}$$

□

2.4.2 Obtaining lower bounds on sub-level set estimates via asymptotics on oscillatory integrals

It turns out that if we have certain a priori asymptotics on oscillatory integrals we can obtain lower bounds on sub-level set estimates. We prove this in our next result below.

Theorem 2.4.2. *Let $I_\lambda = \int_{Q^n} e^{i\lambda\Phi(x)} dx$ and assume for $\lambda > 1$ that a priori the following asymptotics hold*

$$I_\lambda = C\lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon})$$

where C is a constant, and σ, ϵ are real numbers such that $\sigma > 0$ and $\epsilon > 0$ and $\sigma + \epsilon < 1$. Then for the sub-level set $E_\lambda = \{x \in Q^n : |\Phi(x)| < \lambda^{-1}\}$ where $\lambda > 1$, we have that

$$|E_\lambda| \gtrsim \lambda^{-\sigma}.$$

Note. It is easy to see that the above assumptions on σ and ϵ imply that $0 < \sigma < 1$ and $0 < \epsilon < 1$.

However, before we give the proof of the above theorem, we prove the following useful lemma:

Lemma 2.4.3. *Let $0 < \sigma < 1$ and $\xi \in \mathcal{S}(\mathbf{R})$ be a compactly supported positive*

even function. Then

$$\int_0^\infty \frac{\hat{\xi}(s)}{s^\sigma} ds > 0.$$

Proof. Let $\text{supp } \xi = [-a, a]$ for some a such that $a > 0$. It is easy to see that

$$\int_0^\infty \frac{\hat{\xi}(s)}{s^\sigma} ds = 2 \int_0^\infty s^{-\sigma} \int_0^a \cos(2\pi sx) \xi(x) dx ds. \quad (2.4.6)$$

We would now like to apply Fubini's theorem to change the order of integration so that

$$2 \int_0^\infty s^{-\sigma} \int_0^a \cos(2\pi sx) \xi(x) dx ds = 2 \int_0^a \left(\int_0^\infty \frac{\cos(2\pi sx)}{s^\sigma} ds \right) \xi(x) dx.$$

Applying the change of variables $u = 2\pi sx$ we finally obtain that

$$\int_0^\infty \frac{\hat{\xi}(s)}{s^\sigma} ds = 2^\sigma \pi^{\sigma-1} \int_0^a \left(\int_0^\infty \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}} dx.$$

Hence, if we are indeed allowed to apply Fubini's theorem, and furthermore, if we can show that the inner integral is convergent and strictly positive for σ in the range $0 < \sigma < 1$, then we are done.

However, we must be a bit careful, as we cannot apply Fubini's theorem to the right hand side of (2.4.6) straight away as the integral $\int_0^\infty s^{-\sigma} ds$ is not finite. To put ourselves in a situation where it is valid for us to apply Fubini's theorem we must use a limiting argument. Since, when the upper limit in the first integral of (2.4.6) is finite, equal to N say, we can then apply Fubini's theorem and change the order of integration. Then via the Lebesgue dominated convergence theorem we can swap integral sign and limit sign and let $N \rightarrow \infty$ to recover the changed order of integrals with the upper limit in the integral sign being infinite again.

We proceed as follows

$$\begin{aligned}\int_0^\infty s^{-\sigma} \int_0^a \cos(2\pi sx) \xi(x) dx ds &= \lim_{N \rightarrow \infty} \int_0^N s^{-\sigma} \int_0^a \cos(2\pi sx) \xi(x) dx ds \\ &= \lim_{N \rightarrow \infty} \int_0^a \left(\int_0^N \frac{\cos(2\pi sx)}{s^\sigma} ds \right) \xi(x) dx,\end{aligned}$$

and now the change in the order of the integral signs is justified by the application of Fubini's theorem, since we have $\int_0^N \int_0^a |\cos(2\pi sx) \xi(x) s^{-\sigma}| dx ds < \infty$.

We now make the change of variable $u = 2\pi sx$ so that

$$\int_0^a \left(\int_0^N \frac{\cos(2\pi sx)}{s^\sigma} ds \right) \xi(x) dx = (2\pi)^\sigma \int_0^a \left(\int_0^{2\pi xN} \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}} dx,$$

and we let

$$f_N(x) = \left(\int_0^{2\pi xN} \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}}$$

for $0 < x < a$.

Hence, for each N we need to find a function g in L^1 that dominates f_N point-wise.

Moreover, since it is true that

$$|f_N(x)| = \left| \int_0^{2\pi xN} \frac{\cos(u)}{u^\sigma} du \right| \frac{\xi(x)}{x^{1-\sigma}},$$

in order to have any hope of bounding f_N by an L^1 function we need to show that $\int_0^\infty \frac{\cos(u)}{u^\sigma} du$ is finite, as then there will exist a finite $C > 0$ such that for any $M > 0$ we will have $\left| \int_0^M \frac{\cos(u)}{u^\sigma} du \right| \leq C$. Then the obvious choice for our L^1 dominating function g will be

$$g(x) = C \cdot \frac{\xi(x)}{x^{1-\sigma}}$$

and since $0 < \sigma < 1$ we have

$$\|g\|_{L^1} = C \int_0^a \frac{\xi(x)}{x^{1-\sigma}} dx \leq C \int_0^a \frac{dx}{x^{1-\sigma}} = C \cdot \frac{a^\sigma}{\sigma} < \infty,$$

and hence g is indeed in L^1 .

We now consider the integral

$$\int_0^\infty \frac{\cos(u)}{u^\sigma} du \tag{2.4.7}$$

in order to determine whether it is finite and strictly positive. One can easily check that it does indeed converge by integrating by parts twice. Nevertheless, the integrand is of an oscillatory nature, and so, it is not immediately clear as to whether the integral is positive or negative. The answer to this question can be determined by evaluating the integral explicitly by using the complex analytic technique of contour integration, via which one can see that

$$\int_0^\infty \frac{\cos(u)}{u^\sigma} du = \frac{\pi}{2\Gamma(\sigma) \cos(\frac{\sigma\pi}{2})} \tag{2.4.8}$$

where Γ is the Gamma function. Therefore, the answer to our second question, as to whether the integral in (2.4.7) is strictly positive, is now manifestly clear, since σ is such that $0 < \sigma < 1$.

We are now in a position to apply the Lebesgue dominated convergence theorem.

For fixed $0 < x < a$ we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} f_N(x) &= \left(\lim_{N \rightarrow \infty} \int_0^{2\pi x N} \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}} \\ &= \left(\int_0^\infty \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}}, \end{aligned}$$

and so

$$\begin{aligned}\lim_{N \rightarrow \infty} \int_0^a f_N(x) dx &= \int_0^a \lim_{N \rightarrow \infty} f_N(x) dx \\ &= \int_0^a \left(\int_0^\infty \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}} dx.\end{aligned}$$

So that finally

$$\int_0^\infty \frac{\hat{\xi}(s)}{s^\sigma} ds = 2^\sigma \pi^{\sigma-1} \int_0^a \left(\int_0^\infty \frac{\cos(u)}{u^\sigma} du \right) \frac{\xi(x)}{x^{1-\sigma}} dx,$$

and the right hand side of the equality above is finite and strictly positive, thus completing the proof. \square

We are now in a position to prove Theorem 2.4.2.

2.4.3 Proof of Theorem 2.4.2

Proof. Let $\lambda > 0$, by definition $|E_\lambda| = \int_{Q^n} \chi_{E_\lambda}(x) dx$. Construct $\xi \in C_c^\infty(\mathbf{R})$, where $\text{supp } \xi = [-a, a]$ with $0 < a \leq 1$, so that point-wise we have the inequality

$$\chi_{E_\lambda}(x) \geq \xi(\lambda \Phi(x)).$$

Then $|E_\lambda| \geq \int_{Q^n} \xi(\lambda \Phi(x)) dx$. Applying Fubini's theorem together with the fact that $\xi(t) = \int_{-\infty}^\infty \hat{\xi}(s) e^{2\pi i s t} ds$ we have that

$$|E_\lambda| \geq \int_{-\infty}^\infty \hat{\xi}(s) I_{\lambda s} ds$$

where $I_{\lambda s} = \int_{Q^n} e^{2\pi i \Phi(x) \lambda s} dx$.

It is trivial to observe that for any real even function its Fourier transform is real

and even, and so remembering this we also have the following

$$\begin{aligned}
\int_{-\infty}^{\infty} \widehat{\xi}(s) I_{\lambda s} ds &= \int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds + \overline{\int_{-\infty}^0 \widehat{\xi}(s) I_{\lambda(-s)} ds} \\
&= \int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds + \overline{\int_{-\infty}^0 \widehat{\xi}(s) I_{\lambda(-s)} ds} \\
&= \int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds + \int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds \\
&= 2\Re \left(\int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds \right).
\end{aligned}$$

Now if we can show that

$$\int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds = C\lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon}),$$

where C is finite and strictly positive, then we will also have that

$$|E_{\lambda}| \geq C\lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon}),$$

since

$$\Re(C\lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon})) = C\lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon}).$$

We proceed by splitting the region of integration as follows

$$\begin{aligned}
\int_0^{\infty} \widehat{\xi}(s) I_{\lambda s} ds &= \int_0^{\lambda^{-1}} \widehat{\xi}(s) I_{\lambda s} ds + \int_{\lambda^{-1}}^{\infty} \widehat{\xi}(s) I_{\lambda s} ds \\
&= I_1 + I_2.
\end{aligned}$$

(i) Estimate for I_1 :

$$|I_1| \leq \int_0^{\lambda^{-1}} |\widehat{\xi}(s)| ds \leq \|\widehat{\xi}\|_{L^{\infty}} \int_0^{\lambda^{-1}} ds \sim \lambda^{-1} < \lambda^{-\sigma-\epsilon}.$$

Hence $I_1 = O(\lambda^{-\sigma-\epsilon})$.

From the estimate above it is clear now that our next goal when estimating I_2 will be to show that $I_2 = C\lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon})$, where $0 < C < \infty$.

(ii) Estimate for I_2 :

Since $s\lambda > 1$ we can apply the asymptotics we have on $I_{\lambda s}$ to write

$$\begin{aligned} I_2 &= \lambda^{-\sigma} \int_{\lambda^{-1}}^{\infty} \frac{\widehat{\xi}(s)}{s^{\sigma}} ds + \int_{\lambda^{-1}}^{\infty} O((\lambda s)^{-\sigma-\epsilon}) \widehat{\xi}(s) ds \\ &= \lambda^{-\sigma} \int_0^{\infty} \frac{\widehat{\xi}(s)}{s^{\sigma}} ds - \lambda^{-\sigma} \int_0^{\lambda^{-1}} \frac{\widehat{\xi}(s)}{s^{\sigma}} ds + \int_{\lambda^{-1}}^{\infty} O((\lambda s)^{-\sigma-\epsilon}) \widehat{\xi}(s) ds. \end{aligned}$$

Since ξ is real and even, its Fourier transform $\widehat{\xi}$ is also real and even, and so we can immediately apply Lemma 2.4.3 to deduce that $\int_0^{\infty} \frac{\widehat{\xi}(s)}{s^{\sigma}} ds$ is finite and strictly positive, and so we have established the desired constant C that we need in front of $\lambda^{-\sigma}$ in the first term of I_2 . All that is left for us to do now is to show that the remaining two terms in I_2 , which we now call I_3 and I_4 respectively, are both $O(\lambda^{-\sigma-\epsilon})$ and then the proof of the result is complete.

Let us take care of the first term $I_3 = \lambda^{-\sigma} \int_0^{\lambda^{-1}} \frac{\widehat{\xi}(s)}{s^{\sigma}} ds$, we have

$$\begin{aligned} \left| \lambda^{-\sigma} \int_0^{\lambda^{-1}} \frac{\widehat{\xi}(s)}{s^{\sigma}} ds \right| &\leq \lambda^{-\sigma} \int_0^{\lambda^{-1}} \frac{|\widehat{\xi}(s)|}{s^{\sigma}} ds \\ &\leq \lambda^{-\sigma} \|\widehat{\xi}\|_{L^{\infty}} \int_0^{\lambda^{-1}} \frac{ds}{s^{\sigma}} \\ &\lesssim \frac{\lambda^{-1}}{1-\sigma} \\ &\lesssim \lambda^{-\sigma-\epsilon}. \end{aligned}$$

Hence $I_3 = O(\lambda^{-\sigma-\epsilon})$.

Note. It is crucial here that we have the hypothesis $0 < \sigma < 1$.

For the second term $I_4 = \int_{\lambda^{-1}}^{\infty} O((\lambda s)^{-\sigma-\epsilon}) \widehat{\xi}(s) ds$ we have

$$\begin{aligned} \left| \int_{\lambda^{-1}}^{\infty} O((\lambda s)^{-\sigma-\epsilon}) \widehat{\xi}(s) ds \right| &\leq \int_{\lambda^{-1}}^{\infty} |O((\lambda s)^{-\sigma-\epsilon})| |\widehat{\xi}(s)| ds \\ &\leq A \lambda^{-\sigma-\epsilon} \int_{\lambda^{-1}}^{\infty} \frac{|\widehat{\xi}(s)|}{s^{\sigma+\epsilon}} ds. \end{aligned}$$

Now, it is easy to see that $\int_{\lambda^{-1}}^{\infty} \frac{|\widehat{\xi}(s)|}{s^{\sigma+\epsilon}} ds = O(1)$. Indeed, we have

$$\begin{aligned} \int_{\lambda^{-1}}^{\infty} \frac{|\widehat{\xi}(s)|}{s^{\sigma+\epsilon}} ds &\leq \|\widehat{\xi}\|_{L^\infty} \int_{\lambda^{-1}}^1 \frac{ds}{s^{\sigma+\epsilon}} + A' \int_1^{\infty} s^{-(\sigma+\epsilon)-1} ds \\ &= \|\widehat{\xi}\|_{L^\infty} \frac{(1 - \lambda^{(\sigma+\epsilon)-1})}{1 - (\sigma + \epsilon)} + \frac{A'}{\sigma + \epsilon} \\ &\leq \frac{\|\widehat{\xi}\|_{L^\infty}}{1 - (\sigma + \epsilon)} + \frac{A'}{\sigma + \epsilon} \lesssim 1, \end{aligned}$$

and so we conclude that $I_4 = O(\lambda^{-\sigma-\epsilon})$.

Note. Again, it is crucial here that we have the hypothesis $0 < \sigma + \epsilon < 1$.

Hence, putting everything together, we finally conclude our estimate for I_2 , namely that

$$I_2 = C \lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon}).$$

So we may now finally put together our estimates for I_1 and I_2 to deduce that

$$|E_\lambda| \geq C \lambda^{-\sigma} + O(\lambda^{-\sigma-\epsilon}).$$

Now the above implies that we have for some $B > 0$ that

$$|E_\lambda| \geq C \lambda^{-\sigma} - B \lambda^{-\sigma-\epsilon},$$

and so, for $0 < \epsilon < 1$, the conclusion of the theorem follows by always taking λ large enough so that $\lambda > \max(1, (\frac{2B}{C})^{1/\epsilon})$. \square

Chapter 3

Stability of global multilinear sub-level set operator estimates

3.1 Introduction

In this chapter we will endeavour to study the stability of sub-level set estimates in a multilinear setting. In particular, we will investigate what happens to the stability of sub-level set estimates under polynomial transformations of the phase Φ , given that we know an a priori estimate for the corresponding multilinear oscillatory integral $\Lambda_\lambda^{\Phi, K, \pi}$. More precisely, our main goal will be to seek the existence of *global* and *uniform* bounds for a general class of sub-level set operators $S_\delta^{P(\Phi), K, \pi}$, where P is a *normalised* polynomial of bounded degree d , given that we know an a priori estimate for the corresponding multilinear oscillatory integral $\Lambda_\lambda^{\Phi, K, \pi}$.

We will be naturally led to analyse the finer structure of sub-level sets

$$S_{\delta, P} := \{t \in \mathbf{R} : |P(t)| < \delta\},$$

and to obtain bounds for $|S_{\delta, P}|$ that are *global* in t . Furthermore, we will see that

our analysis will obtain for us a way of decomposing the sub-level set $S_{\delta,P}$ so that we can obtain bounds, for the general sub-level set operator $S_{\delta}^{P(\Phi),K,\pi}$, that are *uniform* in the coefficients of the normalised polynomial P .

3.2 Exploring the stability of oscillatory integral and sub-level set estimates

Having seen how oscillatory integral estimates imply sub-level set estimates we would now like to explore this further with regard to certain stability issues. To illustrate this we bring to attention a notable paper of D. H. Phong and E. M. Stein [34]. Now, in this paper the authors obtain sharp and general bounds for oscillatory integral operators on $L^2(\mathbf{R})$ of the form

$$(T_{\lambda}f)(x) = \int_{-\infty}^{\infty} e^{i\lambda\Phi(x,y)}\psi(x,y)f(y)dy,$$

where $\psi \in C_0^{\infty}(\mathbf{R}^2)$ is a smooth cut-off function supported in a small neighbourhood of the origin, and the phase $\Phi(x,y)$ is real-analytic. The sharp bounds for $\|T_{\lambda}\|_{L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})}$ are determined by the *reduced Newton polyhedron* of the phase $\Phi(x,y)$.

The *Newton polyhedron* is a remarkable geometric notion which in the 70's had been shown by A. N. Varčenko [43] to control the apparently unrelated decay rate for the *two dimensional scalar* oscillatory integral with phase $\Phi(x,y)$, confirming earlier hypotheses of Arnold. Before we go on to state the full result of D. H. Phong and E. M. Stein [34]; we will now, for the reader's convenience, and for the purposes of rendering more intelligible certain future comments regarding decay rates, recall the notion of the Newton polyhedron. We will also supply the definition of the *reduced Newton polyhedron* and the *Newton decay rate* as given in [34].

Note. Since we are considering phases of two variables, we will stick to defining the various geometric notions in the context of two dimensions, although the reader should note that the following definitions are exactly the same for a non-constant real analytic function of n variables.

Let Φ be a non-constant real analytic function of two variables defined on a neighbourhood of $0 \in \mathbf{R}^2$, satisfying $\Phi(0) = \nabla\Phi(0) = 0$. Let

$$\Phi(x, y) = \sum_{j,k=0}^{\infty} c_{jk} x^j y^k$$

be the Taylor series expansion of Φ about the origin. Let $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$, and $\text{supp}(\Phi) = \{(j, k) \in \mathbf{N}_0^2 : c_{jk} \neq 0\}$ denote the *Newton support* of Φ . The *Newton polyhedron* $\Gamma_+(\Phi)$ of the Taylor series of Φ is defined as the convex hull in \mathbf{R}_+^2 of

$$\bigcup_{\omega \in \text{supp}(\Phi)} (\omega + \mathbf{R}_+^2).$$

The *reduced Newton polyhedron* is defined in the same way, with this time the vertices $(j, k) \in \text{supp}(\Phi)$ constrained by the additional requirement that $jk \neq 0$.

The edges¹ of the Newton polyhedron are called *Newton diagrams*, and the *Newton diagram* $\Gamma(\Phi)$ of Φ is defined as the union of all of the edges of $\Gamma_+(\Phi)$. The *Newton decay rate* is defined as

$$\gamma = \min_{\ell} \gamma_{\ell}$$

where the index ℓ runs through the boundary lines of the reduced Newton diagram, and $(\gamma_{\ell}^{-1}, \gamma_{\ell}^{-1})$ is the intersection of the line ℓ with the line $j = k$ bisecting the first quadrant. So, with all of these basic definitions out of the way, we are

¹In the case of n dimensions it will be compact faces.

now in a position where we can return to our previous discussion and state the sharp L^2 result of D. H. Phong and E. M. Stein. The result is the following.

Theorem 3.2.1. *[34] Let $\Phi(x, y)$ be a real-analytic phase function. If the support of ψ is sufficiently small, then the operator T_λ is bounded on $L^2(\mathbf{R})$ with the bound*

$$\|T_\lambda\|_{L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})} \leq C|\lambda|^{-\frac{1}{2}\gamma}, \quad (3.2.1)$$

where γ is the Newton decay rate with respect to the reduced Newton diagram. The result (3.2.1) is exact in the sense that if ψ is not zero at the origin, then $\|T_\lambda\|_{L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})} \geq c'|\lambda|^{-\frac{1}{2}\gamma}$, as $|\lambda| \rightarrow \infty$, for some $c' > 0$.

It is worth noting that Theorem 3.2.1 also implies sharp L^2 Sobolev estimates for certain model operators of Radon type that correspond to averaging operators over curves. This is in fact one of the main reasons why the authors considered studying this particular class of oscillatory integral operators of the second kind on $L^2(\mathbf{R})$, see for example D. H. Phong and E. M. Stein [31], [33], and also A. Seeger [38].

We can of course then apply this sharp result to Theorem 2.3.1 for the purpose of obtaining sharp sub-level set estimates. However, we would like to explore things further, and in particular, we would like to see how this relates to stability issues i.e. where one is interested in studying how estimates behave under changes of the phase. Ideally, one would like to develop a calculus for oscillatory integral estimates which behaves well under changes of the phase, but this is difficult to achieve in general².

For instance, if Φ is a real-analytic phase as in Theorem 3.2.1 with Newton decay

²However, we will discuss in more detail in the next chapter a particular simple situation where we are able to make progress on this matter.

rate γ , and we consider, for small $0 < \epsilon < 1$, the *perturbed* phase $P_\epsilon \circ \Phi$, where

$$P_\epsilon(t) = t^2 - \epsilon t,$$

and also the associated oscillatory integral operator $T_\lambda^{P_\epsilon \circ \Phi}$ given by

$$T_\lambda^{P_\epsilon \circ \Phi} f(x) = \int_{-\infty}^{\infty} e^{i\lambda[(\Phi(x,y))^2 - \epsilon\Phi(x,y)]} \psi(x,y) f(y) dy;$$

then the reduced Newton diagram and the Newton decay rate remain unchanged for all $\epsilon > 0$ but undergo a sudden jump at $\epsilon = 0$.

Now, if one seeks a bound for $\|T_\lambda^{P_\epsilon \circ \Phi}\|_{L^2 \rightarrow L^2}$ that is uniform for all $\epsilon \in (0, 1)$, then one would expect the uniform decay rate to be $\frac{\gamma}{4}$, which is the decay rate that one would expect to obtain in the bound for $\|T_\lambda^{\Phi^2}\|_{L^2 \rightarrow L^2}$ after applying Theorem 3.2.1, given that the Newton decay rate of Φ is γ . To illustrate why one would expect the decay rate in the estimate for $\|T_\lambda^{\Phi^2}\|_{L^2 \rightarrow L^2}$ to be $\frac{\gamma}{4}$, we will study the simple phase

$$\Phi(x, y) = cx^{j_0}y^{k_0} + dx^{j_1}y^{k_1},$$

where $c, d \neq 0$, and for each $\ell = 0, 1$ we have (j_ℓ, k_ℓ) such that $j_\ell k_\ell \neq 0$.

Let β_Φ denote the Newton decay rate of Φ for this particular example. The reduced Newton polyhedron in this case is very simple, and, by drawing a picture of it, one can easily see that to obtain β_Φ we must calculate the coordinates of the point of intersection of the line

$$k = -\left(\frac{k_0 - k_1}{j_1 - j_0}\right)(j - j_0) + k_0 \tag{3.2.2}$$

with the line $k = j$; with the coordinates of this point being (β^{-1}, β^{-1}) . Since this is the only point of intersection between the reduced Newton polyhedron and

the line $k = j$, we may then take β_Φ to be β . Hence, setting $k = j = \beta^{-1}$ in equation (3.2.2), one then obtains via some simple algebra that

$$\beta_\Phi = \frac{1 + \left(\frac{k_0 - k_1}{j_1 - j_0}\right)}{k_0 + \left(\frac{k_0 - k_1}{j_1 - j_0}\right) j_0}.$$

Next, turning our attention to Φ^2 , we observe that

$$\Phi^2(x, y) = c^2 x^{2j_0} y^{2k_0} + 2dcx^{j_0+j_1} y^{k_0+k_1} + d^2 x^{2j_1} y^{2k_1}.$$

Again, the reduced Newton diagram in this case is very simple, and it is easy to see from it that to obtain β_{Φ^2} we must now calculate the coordinates of the point of intersection of the line

$$k = -\left(\frac{k_0 - k_1}{j_1 - j_0}\right)(j - 2j_0) + 2k_0 \quad (3.2.3)$$

with the line $k = j$; with the coordinates of this point being $(\beta'^{-1}, \beta'^{-1})$. Since this is the only point of intersection between the reduced Newton polyhedron and the line $k = j$, we may take β_{Φ^2} to be β' . Hence, setting $k = j = \beta'^{-1}$ in equation (3.2.3), one then obtains via some simple algebra that

$$\beta_{\Phi^2} = \left(\frac{1 + \left(\frac{k_0 - k_1}{j_1 - j_0}\right)}{k_0 + \left(\frac{k_0 - k_1}{j_1 - j_0}\right) j_0}\right) \cdot \frac{1}{2},$$

and therefore that $\beta_{\Phi^2} = \frac{1}{2}\beta_\Phi$. Hence, for this example, since the Newton decay rate for Φ is β_Φ , the decay rate in the estimate for $\|T_\lambda^{\Phi^2}\|_{L^2 \rightarrow L^2}$, by applying Theorem 3.2.1, is $\frac{\beta_\Phi}{4}$.

So, we see from this simple example that if the Newton decay rate of Φ is γ , one would expect the decay rate in the estimate for $\|T_\lambda^{\Phi^2}\|_{L^2 \rightarrow L^2}$ to be $\frac{\gamma}{4}$; and hence, if one were to seek a bound for $\|T_\lambda^{P_\epsilon \circ \Phi}\|_{L^2 \rightarrow L^2}$, where $P_\epsilon \circ \Phi = \Phi^2 - \epsilon\Phi$, that is

uniform for all $\epsilon \in (0, 1)$, one would indeed expect to obtain

$$\|T_\lambda^{P_\epsilon \circ \Phi}\|_{L^2 \rightarrow L^2} \leq C|\lambda|^{-\frac{\gamma}{4}} \quad (3.2.4)$$

where the constant C is independent of ϵ . However, such an estimate, as the one above in (3.2.4), cannot be read straight off from Theorem 3.2.1, but perhaps one can go back to the proof and discover this somehow.

On the other hand, we will see that the desired uniform estimates do hold for the corresponding sub-level set operators $S_\delta^{P_\epsilon \circ \Phi}$, which are defined as

$$S_\delta^{P_\epsilon \circ \Phi} f(x) = \int_{\{(x,y) \in K : |(\Phi(x,y))^2 - \epsilon \Phi(x,y)| < \delta\}} f(y) dy,$$

where K is an arbitrary compact set contained in $\mathbf{R} \times \mathbf{R}$. Namely, for every $0 < \epsilon < 1$, we will see later in Section 3.7 of this chapter that

$$\|S_\delta^{P_\epsilon \circ \Phi}\|_{L^2 \rightarrow L^2} \leq C\delta^{\frac{\gamma}{4}}.$$

This in fact follows directly from Theorem 3.2.1 via an application of the forthcoming Corollary 3.6.2.

To begin to see why sub-level sets are more amenable to polynomial changes of the phase, we simply note that, for $\delta > 0$

$$|\{x : |\Phi(x)| < \delta\}| \leq C\delta^\eta \quad (3.2.5)$$

immediately implies that

$$|\{x : |(\Phi(x))^k| < \delta\}| \leq C\delta^{\eta/k} \quad (3.2.6)$$

for trivial algebraic reasons. The corresponding statement for oscillatory integrals

is much less clear.

However, if we simply try to shift the phase Φ by a very large constant we can affect the sub-level set estimate quite drastically. The heart of the matter lies in the fact that sub-level set estimates are sensitive to the neighbourhoods of the zeros of Φ . That is, if we shift Φ by a very large constant we alter its zeros, and this alters their neighbourhoods to the extent that they may no longer intersect the sub-level set of the new shifted phase. The following example illustrates this point very clearly.

Example 3.2.2. Let $\Phi(x) = x^2$ and $P(t) = t \pm c$ with $c = 100$. We consider the sub-level set

$$\{x \in [0, 1] : |P(\Phi(x))| < \delta\}. \quad (3.2.7)$$

We see that with $c = 100$ the new phase $P \circ \Phi$ no longer has any zeros at all and so our sub-level set is the empty set. When $c = -100$, even though $P \circ \Phi$ does have two zeros, namely $x = \pm 10$; their neighbourhoods are simply too far away from the interval $[0, 1]$ to even intersect it at all and to thus contribute to the measure of the sub-level set, and so again we are left with the sub-level set being the empty set. Hence, in both cases the measure of the sub-level set in (3.2.7) is zero, whereas we have

$$|\{x \in [0, 1] : |\Phi(x)| < \delta\}| = \delta^{1/2}.$$

On the other hand, oscillatory integral estimates are not affected by translations of the phase. This is because the crucial factor responsible for determining the behaviour of oscillatory integral estimates is the derivatives of the phase Φ , and thus its critical points; and the critical points do not change under constant shifts of the phase. And so, if we also know that the estimate in (3.2.5) arises from an a priori oscillatory integral estimate via the folklore technique, the task then

becomes a matter of applying the folklore technique with the phase replaced by $\Phi(x) \pm c$.

Having considered constant shifts as well as powers of the phase Φ , we can naturally put the two transformations together to go one step further in generality to consider the case when we have a centred monomial function of Φ , namely $(\Phi(x) - c)^k$. In light of the considerations given above, and provided that an estimate such as the one in (3.2.5) arises from an a priori oscillatory integral estimate via the folklore technique, it is clear that we can also obtain the estimate for the measure of the sub-level set corresponding to the phase $(\Phi(x) - c)^k$.

The considerations we have made so far provide the impetus for us to consider sub-level set operators with general polynomial phases $P(\Phi)$ where P is a polynomial of degree d i.e.

$$P(t) = c_0 + c_1 t + \dots + c_d t^d.$$

Moreover, since we will want to utilise the folklore technique, it will be our goal to also construct a procedure that will transform our more general sub-level set operator into one having a phase which is of a monomial kind. We will consider the problem in further generality by formulating it in a multilinear setting, and our prime goal will be to obtain for the sub-level set operator a *global* bound that is *uniform* in the coefficients of the polynomial P . We will obtain our estimate for the class of *normalised* polynomials P .

3.3 Global sub-level set estimates in the multilinear setting

We will now consider a phase Φ such that

$$\Phi : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}.$$

The general *multilinear oscillatory integral operator* (or L -linear form) is defined by the expression

$$\Lambda_{\lambda}^{\Phi, K, \pi}(f_1, \dots, f_L) = \int_K e^{i\lambda\Phi(x)} \prod_{j=1}^L f_j(\pi_j(x)) dx \quad (3.3.1)$$

where $K \subset \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L}$ is a compact set, $x \in \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L}$, and $\pi_j : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}^{m_j}$ are general mappings for each $j \in \{1, \dots, L\}$ that belong to the set $\pi = \{\pi_1, \dots, \pi_L\}$.

The integral $\Lambda_{\lambda}^{\Phi, K, \pi}$ in (3.3.1) is well defined if all f_j belong to $L^{\infty}(\mathbf{R}^{m_j})$ and it satisfies

$$|\Lambda_{\lambda}^{\Phi, K, \pi}(f_1, \dots, f_L)| \leq C \prod_{j=1}^L \|f_j\|_{L^{\infty}(\mathbf{R}^{m_j})}.$$

We then define the corresponding *multilinear sub-level set operator* by the expression

$$S_{\delta}^{\Phi, K, \pi}(f_1, \dots, f_L) = \int_{\{x \in K : |\Phi(x)| < \delta\}} \prod_{j=1}^L f_j(\pi_j(x)) dx. \quad (3.3.2)$$

It is useful to consider such an operator as the one given in (3.3.2), for once we know the mapping properties of the sub-level set operator, we can deduce, as a simple consequence of choosing the appropriate characteristic functions, a bound for the measure of the sub-level set $\{x \in K : |\Phi(x)| < \delta\}$.

In fact, one can essentially encapsulate many of the problems in harmonic analysis by studying operators such as the ones given in (3.3.1) and (3.3.2); and the oscillatory integral operator given in (3.3.1) is a generalisation of oscillatory integrals operators of the first and second kind. Moreover, any non-trivial estimates on the oscillatory integral operator given in (3.3.1) will automatically transfer to estimates for oscillatory integrals of the first kind.

For example, when $L = 2$, and $n_1 = n$ and $n_2 = n - 1$, setting $f = f_1$, $g = f_2$, with $x = x_1$, $y = x_2$, and considering $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, we have the bilinear form

$$\Lambda_{\lambda}^{\Phi, K, \pi}(f, g) = \int_K e^{i\lambda\Phi(x, y)} f(x)g(y) dx dy$$

that we encountered in the guise of (1.8.3) in the restriction problem that was formulated in Section 1.8 of Chapter 1; and taking $f_j \equiv 1$ for each $j \in \{1, \dots, L\}$, we obtain

$$\Lambda_{\lambda}^{\Phi, K, \pi}(f_1, \dots, f_L) = \int_K e^{i\lambda\Phi(x)} dx.$$

Multilinear oscillatory integral forms such as (3.3.1), where in particular, the mappings $\pi_j : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}^{m_j}$ are taken to be surjective linear transformations for $1 \leq j \leq L$, and the phase Φ is taken to be a real-valued polynomial, have been studied in J. Bennett, A. Carbery, M. Christ, and T. Tao [3], and M. Christ, X. Li, T. Tao, and C. Thiele [8]. It is worth mentioning at this point that nonoscillatory inequalities of the form

$$\int \prod_j |f \circ \pi_j| \lesssim \prod_j \|f_j\|_{L^{p_j}}$$

have been studied in J. Bennett, A. Carbery, M. Christ, and T. Tao [2], [3], and one can in fact find in Section 3 of [3] some applications of results concerning

these types of nonoscillatory inequalities to the study of multilinear oscillatory forms such as (3.3.1). In addition, at the end of Section 2 of [3], the authors also bring to the readers attention that further applications to oscillatory integrals will appear in a forthcoming paper of M. Christ and J. Holmer [10].

Returning to the subject of multilinear oscillatory forms, let us focus our attention for a moment on the paper of M. Christ, X. Li, T. Tao, and C. Thiele [8], and so, for the purposes of furthering our discussion, we recall the following two basic questions which are posed by the authors in this paper.

Under what conditions do there exist $\alpha > 0$ and $C < \infty$ such that for all functions $f_j \in L^\infty(\mathbf{R}^{m_j})$,

$$|\Lambda_\lambda^{\Phi, K, \pi}(f_1, \dots, f_L)| \leq C|\lambda|^{-\alpha} \prod_{j=1}^L \|f_j\|_{L^\infty(\mathbf{R}^{m_j})} \quad (3.3.3)$$

for all $\lambda \in \mathbf{R}$?

Under what conditions does there exist a function Θ satisfying $\Theta(\lambda) \rightarrow 0$ as $|\lambda| \rightarrow \infty$ such that for all functions $f_j \in L^\infty(\mathbf{R}^{m_j})$,

$$|\Lambda_\lambda^{\Phi, K, \pi}(f_1, \dots, f_L)| \leq \Theta(\lambda) \prod_{j=1}^L \|f_j\|_{L^\infty(\mathbf{R}^{m_j})} \quad (3.3.4)$$

for all $\lambda \in \mathbf{R}$?

Oscillatory integral inequalities of the type given in (3.3.3) have been extensively studied in the bilinear case, where $L = 2$. Here one is dealing with bilinear forms $\langle T_\lambda(f_1), f_2 \rangle$, and as the reader will recall, the associated linear operators T_λ are commonly known in the literature as oscillatory integrals of the second kind; and a simple necessary and sufficient condition for (3.3.3) to hold, with some unspec-

ified exponent, is known, see [41].

An obvious necessary condition for (3.3.3) is that Φ should be, to use the terminology of [8], *nondegenerate* relative to $\pi = \{\pi_1, \dots, \pi_L\}$. This means that Φ cannot be expressed as a linear combination of measurable functions $\varphi_j \circ \pi_j$ for any measurable functions φ_j . Indeed, if it is the case that $\Phi = \sum_j \varphi_j \circ \pi_j$ for any measurable functions φ_j , then for $f_j = e^{-i\lambda\varphi_j} \in L^\infty$, one then has

$$e^{i\lambda\Phi(x)} \prod_j f_j(\pi_j(x)) \equiv 1,$$

and there is consequently no decay.

Of course, as the authors of [8] point out, one can formulate many numerous variants of the question concerning (3.3.4). For instance, one could also ask the question if (3.3.4) holds with Θ being a *specific* function of $(1 + |\lambda|)$, rather than asking as above whether there just exists *some* function Θ tending to zero as $|\lambda| \rightarrow \infty$ for which (3.3.4) holds. Moreover, whenever (3.3.3) does hold, one can also ask what the optimal power of α might be in (3.3.3). However, in [8], the authors choose to focus on the formulation given in (3.3.3), as it is the one that is most relevant to their applications concerning *multilinear singular integral operators*.

In fact, oscillatory forms such as the one given in (3.3.1), where in particular, the mappings $\pi_j : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}^{m_j}$ are taken to be surjective linear transformations for $1 \leq j \leq L$, and the phase Φ is taken to be a real-valued polynomial, arise in the study of *multilinear singular integral operators*, and one of the main purposes of [8] is to establish bounds for such operators. More precisely, for any real-valued polynomial $P(x, t)$ of degree d , the authors first consider the following singular integral operator

$$T(f, g)(x) = \int_{-\infty}^{\infty} e^{iP(x, t)} f(x - t) g(x + t) t^{-1} dt,$$

where it is initially assumed that $f, g \in C_0^1$, the class of all continuously differentiable functions having compact supports, and the integral is taken in the principal-value sense. One of the main goals of [8] is to establish the following L^p bounds for the above operator.

Theorem 3.3.1. *For any exponents $p_1, p_2, q \in (0, \infty)$ such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_1, p_2 > 1$, and $q > \frac{2}{3}$, and any degree $d \geq 1$, there exists $C < \infty$ such that $\|T(f, g)\|_{L^q} \leq C \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$ for all $f, g \in C_0^1$, uniformly for all real-valued polynomials P of degrees at most d .*

The cases $d = 0, 1, 2$ were previously known, and the case $d = 0$ is the celebrated theorem of M. Lacey and C. Thiele [23]. Theorem 3.3.1 is actually a bilinear analogue of a theorem of F. Ricci and E. M. Stein [36], who proved L^p estimates for linear operators

$$f \mapsto \int e^{iP(x, t)} f(x - t) K(t) dt,$$

for arbitrary real-valued polynomials P and Calderón-Zygmund kernels K . In fact, in [8], the authors actually obtain Theorem 3.3.1 as a special case of an even more general result which they also prove in this paper. The more general theorem, which is about *multilinear singular integral operator L^p bounds*, is in fact a synthesis of the nonoscillatory case $P \equiv 0$, which was treated in C. Muscalu, T. Tao, and C. Thiele [24], and the new results for nonsingular oscillatory integrals contained in [8].

We shall now take the time to briefly look at some examples of (3.3.1) when the oscillatory factor is suppressed i.e. when $\Phi \equiv 0$.

Example 3.3.2. Let $\pi_j = i_d$ for each $j \in \{1, \dots, L\}$, and set $n_1 = n$ with $n_j = 0 \ \forall j \neq 1$, then we have by the general Hölder inequality that

$$\Lambda^{K,\pi}(f_1, \dots, f_L) = \left| \int_K \prod_{j=1}^L f_j(x) dx \right| \leq \prod_{j=1}^L \|f_j\|_{L^{p_j}(\mathbf{R}^n)},$$

for all $f_j \in L^{p_j}(K)$ and $\sum_{j=1}^L \frac{1}{p_j} = 1$.

Example 3.3.3. Let $L = 3$, set $n_1 = n_2 = n$ with $n_3 = 0$, and call $x_1 = x$, $x_2 = y$. Let $K = \mathbf{R}^{2n}$ and consider the mappings π_1, π_2, π_3 , given by

$$\begin{aligned} \pi_1(x, y) &= x \\ \pi_2(x, y) &= x - y \\ \pi_3(x, y) &= y \end{aligned}$$

we then have

$$\begin{aligned} \Lambda^{K,\pi}(f_1, f_2, f_3) &= \int_{\mathbf{R}^{2n}} f_1(x) f_2(x - y) f_3(y) dx dy \\ &= \int_{\mathbf{R}} f_1(x) (f_2 * f_3)(x) dx. \end{aligned}$$

Applying Hölder's and Young's inequalities gives us that

$$|\Lambda^{K,\pi}(f_1, f_2, f_3)| \leq \|f_1\|_{L^{p_1}(\mathbf{R})} \|f_2\|_{L^{p_2}(\mathbf{R})} \|f_3\|_{L^{p_3}(\mathbf{R})}$$

for $f_j \in L^{p_j}(\mathbf{R})$ for each $j = 1, 2, 3$, where $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2$.

Example 3.3.4. Let $L = n$, and set $n_j = 1$ for each $j = 1, \dots, L$. Let $K = \mathbf{R}^n$ and consider π_j such that $\pi_j : \mathbf{R}^n \rightarrow \mathbf{R}^{n-1}$ is given by $\pi_j(x) = (x_1, \dots, \hat{x}_j, \dots, x_n)$ for each $j = 1, \dots, n$. Here the notation $\hat{}$ denotes omission.

The operator under consideration is then

$$\Lambda^{K,\pi}(f_1, \dots, f_L) = \int_{\mathbf{R}^n} f_1(x_2, \dots, x_n) f_2(x_1, x_3, \dots, x_n) \dots f_n(x_1, \dots, x_{n-1}) dx_1 \dots dx_n.$$

One then has by the classical Loomis-Whitney inequality that

$$\Lambda^{K,\pi}(f_1, \dots, f_L) \leq \prod_{j=1}^n \|f_j\|_{L^{n-1}(\mathbf{R}^{n-1})} \quad (3.3.5)$$

for all $f_j \in L^{n-1}(\mathbf{R}^{n-1})$.

Remark. One can actually view the Loomis-Whitney inequality as an n -parameter isoperimetric inequality, and in fact the classical isoperimetric inequality in \mathbf{R}^n can be easily derived from it (albeit not with the sharp constant depending on n). If one looks at the original paper of L. H. Loomis and H. Whitney [27], one sees that this was the main reason that the authors originally considered inequalities of the form given by (3.3.5).

In this chapter we will of course be focusing our attention on obtaining global bounds on multilinear sub-level set operators as defined in (3.3.2), where instead the phase Φ is now replaced by a new phase $P(\Phi)$, P being a real-valued polynomial, that are uniform in the coefficients of the polynomial P . However, before we go on to state our main goals in a more precise manner, it is pertinent at this point, for the purpose of highlighting some of the similarities and differences between our own goals and what has been recently done by others involved in our field of study, to bring to the readers attention a current paper of M. Christ [9]. We also note that this paper contains similar themes to that of [8], only this time the emphasis is on multilinear sub-level set bounds.

In the same way as before, the author of [9] takes for each j such that $1 \leq j \leq L$ the mappings $\pi_j : \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L} \rightarrow \mathbf{R}^{m_j}$ to be surjective linear transformations,

and the phase Φ is taken to be a real-valued polynomial. Let B denote any compact subset of $\subset \mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L}$, and let $\epsilon > 0$ be given. The author then considers the following sub-level sets

$$E_\epsilon(\Phi, g_1, \dots, g_L) = \{y \in B : |\Phi(y) - \sum_{j=1}^L g_j(\pi_j(y))| < \epsilon\}, \quad (3.3.6)$$

where $g_j : \mathbf{R}^{m_j} \rightarrow \mathbf{R}$ are arbitrary Lebesgue measurable functions which are finite almost everywhere.

The aim here is to study upper measure bounds of the form

$$|E_\epsilon(\Phi, g_1, \dots, g_L)| \leq \Theta(\epsilon) \quad (3.3.7)$$

which are uniform over all measurable functions g_j , with $\Theta(\epsilon) \rightarrow 0$. Such bounds would be implied by conjectured multilinear oscillatory integral inequalities. For instance, if a real-valued measurable function Φ satisfies the inequality in (3.3.3), then there is an upper bound for the measures of these sub-level sets, of the form

$$|E_\epsilon(\Phi, g_1, \dots, g_L)| \leq AC\epsilon^\alpha \quad (3.3.8)$$

uniformly for all measurable functions g_j . If instead, however, Φ satisfies the inequality in (3.3.4), then there is a corresponding weakened version of (3.3.8) in which the right-hand side is replaced by a function of ϵ which tends to zero as $\epsilon \rightarrow 0$.

Both of the above implications follow easily by applying the well known folklore procedure, which we have included in Section 2.3 in the form of Theorem 2.3.1. Moreover, because of this connection with multilinear oscillatory integral operators, the sets E_ϵ given in (3.3.6) are appropriately called multilinear sub-level sets.

Whereas only quite restricted classes of polynomials were treated in [8], the main goal of [9] is to establish weak sub-level set bounds, that is, bounds such as the ones presented in (3.3.7), for *all* polynomials satisfying the natural nondegeneracy hypothesis of [8] as well as an additional rationality hypothesis. The analysis involves an alternative notion which is termed *finitely witnessed nondegeneracy*, and relies on a generalisation of Szemerédi's theorem due to H. Furstenberg and Y. Katznelson [18].

Our goal, however, will be to investigate what happens to the stability of sub-level set estimates under polynomial transformations of the phase Φ , given that we know an a priori estimate for the corresponding multilinear oscillatory integral $\Lambda_\lambda^{\Phi, K, \pi}$. It will therefore be our intention to study the more general multilinear sub-level set operator

$$S_\delta^{P(\Phi), K, \pi}(f_1, \dots, f_L) = \int_{\{x \in K : |P(\Phi(x))| < \delta\}} \prod_{j=1}^L f_j(\pi_j(x)) dx,$$

where P is a polynomial of degree d . In fact, our prime aim will be to seek the existence of *global* and *uniform* estimates for the operator $S_\delta^{P(\Phi), K, \pi}$.

By scaling, we need only consider *normalised* polynomials P , and by a normalised polynomial we simply mean a polynomial P where $\|P\| = 1$ with respect to some suitably chosen norm $\|\cdot\|$. The choice of norm in this case is actually irrelevant as we are working in a finite dimensional vector space, and as a result of this all norms are equivalent. However, since we are working over the finite dimensional space of polynomials of degree d , whose elements are of the form $P(t) = \sum_{m=0}^d a_m t^m$, we will take the norm $\|P\| = \max_{0 \leq m \leq d} |a_m|$.

In previous sections of this chapter, we have seen that provided we have a priori oscillatory integral estimates, then we can generally obtain estimates on sub-

level set operators by using the folklore technique; however, the phase in the sub-level set must be a centred monomial. For example, the uniform estimate $|\{x \in K : |\Phi(x) - c| < \eta\}| \leq B\eta^a$ arises from a scalar oscillatory integral estimate with real-valued Φ as the phase, say. Now since it is our aim to study the mapping properties of the sub-level set operator associated to the sub-level set

$$\{x \in K : |P(\Phi(x))| < \delta\},$$

where K is a compact set in $\mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_L}$ and $\Phi : K \rightarrow \mathbf{R}$, we will need to find some way of reducing our more general sub-level set operator to that of one having a phase which is essentially monomial in nature.

We first make the following observation

$$\{x \in K : |P(\Phi(x))| < \delta\} = K \cap \Phi^{-1}(\mathbf{R} \cap \{t : |P(t)| < \delta\}), \quad (3.3.9)$$

and so, we see that we will need to analyse in detail the structure of the global sub-level sets $S_{\delta,P} := \{t \in \mathbf{R} : |P(t)| < \delta\}$ to reduce ourselves to a phase which is essentially monomial in nature. However, it is pertinent at this point to note that a uniform bound for the sub-level set $\{x \in K : |P(\Phi(x))| < \delta\}$, for normalised polynomials P , does not follow from an a priori uniform global sub-level estimate

$$|\{t \in \mathbf{R} : |P(t)| < \mu\}| < A\mu^b, \quad (3.3.10)$$

for normalised P ; starting with the identity in (3.3.9), from which one can immediately see that, even if it is true that a uniform global estimate such as (3.3.10) holds for all $\mu > 0$ and for some $b > 0$, there is still no immediate formal implication that enables one to use (3.3.10) in order to deduce a uniform bound for the sub-level set $K \cap \Phi^{-1}(\{t \in \mathbf{R} : |P(t)| < \delta\})$.

Nevertheless, matters being what they are with regard to the insufficiency of merely possessing uniform global estimates for $|S_{\mu,P}|$, it is important to remark that obtaining a uniform estimate for the global sub-level set $S_{\mu,P}$, via the process of analysing its finer structure, is still a good place for us to start our investigations. For we will see that the estimate itself will aid our intuition as to what we should expect the true uniform bound for the sub-level set $K \cap \Phi^{-1}(\{t \in \mathbf{R} : |P(t)| < \delta\})$ to be, and moreover, the very analysis itself will provide a significant step towards proving that this bound holds globally over all \mathbf{R} in the t variable.

In fact, estimates such as the one given above in (3.3.10) are well known if t is restricted to a normalised interval, say $[-1, 1]$; in which case one can easily deduce, by using the same technique as in the paper of A. Carbery, F. Ricci and J. Wright [5], that some derivative of P is uniformly bounded below when P is normalised. One is then in a position to apply Proposition 1.4.1 to obtain the uniform sub-level set bound

$$|[-1, 1] \cap S_{\delta,P}| \leq A_d \delta^{1/d} \quad (3.3.11)$$

where $d = \deg(P)$ and $\delta < 1$. Moreover, if we take the time to formulate the appropriate conditions on the coefficients of our polynomial, then a slightly more refined analysis yields for us a global in $\delta > 0$ estimate which we prove in the lemma below.

Lemma 3.3.5. *For each k such that $0 \leq k \leq d$ and for every $\epsilon > 0$, there exists an $\eta = \eta(\epsilon, k) > 0$ so that for any real polynomial $Q(t) = \sum_{m=0}^d b_m t^m$ with $|b_k| \geq \epsilon$, and $|b_m| \leq \eta$ for each m such that $k+1 \leq m \leq d$, we have for every $\delta > 0$ the estimate*

$$|[-1, 1] \cap S_{\delta,Q}| \leq A_{d,\epsilon} \delta^{1/k}.$$

Remark. Our main interest in Lemma 3.3.5 is its implication for normalised polynomials Q . When $k = 0$, $\delta^{1/k}$ is simply interpreted as 0 if δ is small and ∞ if δ is large; and it can be seen from the proof what the cutoff between small and large is in this context.

Proof. We will go through the two cases $k = d, d - 1$ explicitly for the benefit of preparing the way for the case of general k in the range $0 \leq k \leq d$, and then we will prove the result for general k .

When $k = d$, the hypotheses on the coefficients imply that $|b_d| \geq \epsilon$, and, in this special case there is no need to consider finding the appropriate η as $\{m : k + 1 \leq m \leq d\} = \emptyset$. We then see that $|Q^{(d)}(t)| \geq |b_d|d! \geq \epsilon d!$ on the interval $[-1, 1]$, well, we actually see that this bound from below on the d -th derivative holds over the whole of \mathbf{R} . Hence, for every $\delta > 0$, the estimate $[-1, 1] \cap S_{\delta, Q} \leq (\epsilon d!)^{-1/d} \delta^{1/d}$, then follows by applying Proposition 1.4.1.

The remaining cases are all very similar to each other, and so we will only go through the case $k = d - 1$ explicitly in order to set the scene for the case of general k lying in the range $0 \leq k \leq d - 1$. When $k = d - 1$, the hypotheses on the coefficients imply that $|b_{d-1}| \geq \epsilon$, and now our task is to find the appropriate η for which having $|b_d| \leq \eta$ will imply for every $\delta > 0$ the corresponding estimate of $[-1, 1] \cap S_{\delta, Q} \leq A_{d, \epsilon} \delta^{1/d-1}$, for some absolute constant $A_{d, \epsilon}$ yet to be determined. We calculate $Q^{(d-1)}(t)$ and observe that

$$\begin{aligned} |Q^{(d-1)}(t)| &= |d!b_d t + (d-1)!b_{d-1}| \\ &\geq (d-1)!\epsilon - d!\eta \end{aligned}$$

provided t lies in the interval $[-1, 1]$. Hence, by choosing $\eta = \frac{\epsilon}{2d}$, we have

$|Q^{(d-1)}(t)| \geq \frac{(d-1)! \epsilon}{2}$ for every $t \in [-1, 1]$, and so, for every $\delta > 0$, the estimate

$$|[-1, 1] \cap S_{\delta, Q}| \leq \left(\frac{\epsilon(d-1)!}{2} \right)^{-1/d-1} \delta^{1/d-1},$$

follows by applying Proposition 1.4.1.

The way for the general case of k satisfying $0 \leq k \leq d-1$ is now clear. For general k the hypotheses on the coefficients imply that $|b_k| \geq \epsilon$, and our goal now is to find the appropriate η for which having $|b_m| \leq \eta$ for every m in the range $k+1 \leq m \leq d$, will imply a lower bound for the k -th derivative of Q in the whole interval $[-1, 1]$. We first calculate the k -th derivative of Q , for which we obtain $Q^{(k)}(t) = \sum_{m=k}^d b_m c_m t^{m-k}$ where $c_m = m(m-1) \cdots (m-k+1)$, and we observe that $c_m \leq d!$ for every m in the range $k \leq m \leq d$. We observe next that

$$\begin{aligned} |Q^{(k)}(t)| &\geq k!|b_k| - \sum_{m=k+1}^d c_m |b_m| |t|^{m-k} \\ &\geq k!\epsilon - \eta \sum_{m=k+1}^d c_m \\ &\geq k!\epsilon - \eta(d-k)d! \end{aligned}$$

provided t lies in the interval $[-1, 1]$. Hence, by choosing $\eta = \frac{k!\epsilon}{2(d-k)d!}$, we have $|Q^{(k)}(t)| \geq \frac{k!\epsilon}{2}$ for every $t \in [-1, 1]$, and so, for every $\delta > 0$, the estimate

$$|[-1, 1] \cap S_{\delta, Q}| \leq \left(\frac{\epsilon k!}{2} \right)^{-1/k} \delta^{1/k},$$

follows by applying Proposition 1.4.1, thus completing the proof for the case of general k .

□

Remark. In the above result it was crucial that for each k such that $0 \leq k \leq d-1$

and for every $\epsilon > 0$, we had the freedom to choose our $\eta = \eta(\epsilon, k)$ so that we could obtain bounds from below on the k -th derivative of our polynomial Q .

Let us return now to the discussion concerning the possibility of *global in t* sub-level set bounds for normalised polynomials. Now, since we are considering smooth phases Φ such that $\Phi : K \rightarrow \mathbf{R}$, it is more important for us to have global in t estimates instead of merely global in δ estimates, as it would be somewhat artificial to impose that the image $\Phi(K)$ should be restricted to the interval $[-1, 1]$. This consideration therefore naturally leads us to ask whether uniform estimates such as (3.3.11), and the one obtained in Lemma 3.3.5, still hold if t is no longer restricted to the normalised interval $[-1, 1]$.

The answer is of course a resounding yes if the normalising coefficient of our polynomial P occurs in the top term; that is, if P is monic. For then the d -th derivative of P is uniformly bounded below and thus we can use Proposition 1.4.1. However, if we consider the simple example $P(t) = \epsilon t^2 - t$, where $0 < \epsilon < 1$, then one easily sees that no such derivative bounds are available if t is unrestricted, and thus attempting to achieve *global in t* sub-level set bounds for normalised polynomials, via uniform bounds for some derivative of the phase, fails outside the monic case. It will prove useful for us to now examine this simple example in more detail in order to first illustrate, and to also get a feel for the general case, as to what terms in the polynomial P will actually control matters globally in t when bounding $|S_{\delta, P}|$.

Furthermore, if we recall that our main goal in the end is to seek a bound for $|\{x \in K : |P(\Phi(x))| < \delta\}|$ that is uniform globally for all normalised polynomials, then it will of course also be of benefit to us, given the uniform data $|\{x \in K : |\Phi(x) - c| < \eta\}| \leq B\eta^a$ which we know already can arise from a scalar oscillatory integral estimate with real-valued Φ as the phase, say, to estimate the

bound for the sub-level set $\{x \in K : |P(\Phi(x))| < \delta\}$ in the case of this particular normalised polynomial $P(t) = \epsilon t^2 - t$.

Now, before we go any further with the details of this example, we note in passing, what we will see in a moment, that we could also just as well assume a priori the uniform data $|\{x \in K : |\Phi(x) - c| < \eta\}| \leq B\eta^a$ and go on to bound the sub-level set $\{x \in K : |P(\Phi(x))| < \delta\}$. However, it was precisely because of the fact that oscillatory integral estimates remain invariant under translations of the phase by any constant, that we considered studying sets such as $\{x \in K : |\Phi(x) - c| < \eta\}$ in the first place.

Example 3.3.6. Consider the normalised polynomial $P(t) = \epsilon t^2 - t$. There are two roots $t = 0$ and $t = 1/\epsilon$. This suggests that we employ a decomposition of the real line given by $\mathbf{R} = I_1 \cup I_2 \cup I_3$ where I_1 is a neighbourhood of 0, given by $(-\frac{1}{2\epsilon}, \frac{1}{2\epsilon})$ on which $|P(t)| \sim |t|$, I_2 is a neighbourhood of $1/\epsilon$, given by $(\frac{1}{2\epsilon}, \frac{3}{2\epsilon})$, on which $P(t) \sim |t - 1/\epsilon|$, and I_3 is the complement of $I_1 \cup I_2$, given by $\mathbf{R} \setminus (\{t : t < -\frac{1}{2\epsilon}\} \cup \{t : t > \frac{3}{2\epsilon}\})$, on which $P(t) \sim \epsilon|t|^2$ or (equivalently) $P(t) \sim \epsilon|t - 1/\epsilon|^2$.

Thus $\{t \in \mathbf{R} : |P(t)| < \delta\} \sim$

$$\{|t| \lesssim \min(\delta, 1/\epsilon)\} \cup \{|t - 1/\epsilon| \lesssim \min(\delta, 1/\epsilon)\} \cup \{1/\epsilon \lesssim |t| \lesssim \sqrt{(\delta/\epsilon)}\}.$$

Such a decomposition on the real line then gives rise to a decomposition of the set K given by $K = K_1 \cup K_2 \cup K_3$ so that $\{x \in K : |P(\Phi(x))| < \delta\} \sim$

$$\{|\Phi(x)| \lesssim \min(\delta, 1/\epsilon)\} \cup \{|\Phi(x) - 1/\epsilon| \lesssim \min(\delta, 1/\epsilon)\} \cup \{1/\epsilon \lesssim |\Phi(x)| \lesssim \sqrt{(\delta/\epsilon)}\}.$$

And so, by considering the two separate cases as to whether $1/\epsilon \lesssim \sqrt{(\delta/\epsilon)}$ or $1/\epsilon \gtrsim \sqrt{(\delta/\epsilon)}$, we see that the uniform estimate $|\{x \in K : |\Phi(x) - c| < \eta\}| < B\eta^a$

leads to $|\{x \in K : |P(\Phi(x))| < \delta\}| < B'\delta^a$, which is uniform for all $\delta > 0$. In particular the argument also gives us, uniformly for all $\delta > 0$ and $\epsilon \in \mathbf{R}$, a *global* sub-level set bound $|\{t \in \mathbf{R} : |P(t)| < \delta\}| < 10\delta$.

Hence, we see that in this example it is the linear term which actually controls matters globally. In fact, if one also considers the normalised polynomial $P(t) = t^2 - \epsilon t$, and one employs the exact same argument as given above, one can deduce that in this particular case it is the quadratic term which controls matters globally. This observation regarding $P(t) = t^2 - \epsilon t$ is of course not surprising given our previous comments concerning monic polynomials of degree d possessing d -th derivatives that are uniformly bounded from below. Nevertheless, one can follow the same details in this case to see that one obtains a global sub-level set bound of $\delta^{1/2}$ which is uniform for all $\delta > 0$ and $\epsilon \in \mathbf{R}$. Consequently, we see that we will have to set about constructing a technical decision process for determining what controls matters globally in the general polynomial case.

However, it is worthwhile to go back for a moment to the issue of polynomials possessing certain derivatives which are uniformly bound below. We have already observed that if our polynomial P is monic then the d -th derivative of P is uniformly bounded below, and thus we can use Proposition 1.4.1 to trivially obtain uniform global bounds for $|S_{\delta,P}|$. However, if $P(t) = \epsilon t^d \pm t^{d-1} + \dots$ where ϵ is small, then any uniform estimate for $|S_{\delta,P}|$ would imply the same estimate for $|S_{\delta,Q}|$ when $Q(t) = \pm t^{d-1} + \dots$ and so we may expect to achieve a uniform global estimate of $\delta^{1/(d-1)}$ for $|S_{\delta,P}|$ by obtaining a uniform estimate for the $(d-1)$ -st derivative of P . This approach is too naive of course as the simple example $P(t) = \epsilon t \pm 1$ shows; the 0-th derivative in this case has no uniform bound and there are in fact no uniform, global in t , estimates for this example, even if δ is restricted to be small.

The fact that this is the only counterexample in this case is perhaps somewhat surprising. More precisely, our analysis below will show that if $P(t) = \epsilon t^d \pm t^{d-1} + \dots$ and the degree d satisfies the lower bound $d \geq 2$, then there are uniform in t estimates; in fact if $|\epsilon| \leq \rho$, for some small $\rho = \rho_d$, which we will determine precisely from the analysis, then we have the uniform estimate $|S_{\delta,P}| \leq A_d \delta^{1/(d-1)}$ and this is valid for any $\delta > 0$. Observe, however, that if $\rho_d < |\epsilon| \leq 1$, then we obtain instead the uniform estimate $|S_{\delta,P}| \leq A_d \delta^{1/d}$, which is valid for any $\delta > 0$, by using the d -th derivative.

Trivial counterexamples continue to persist when the coefficient of the normalised polynomial which is 1 occurs in other places; for example the polynomial $P(t) = \epsilon t^2 - 1$ is normalised in the $(d-2)$ -nd place and one can see that no uniform estimates hold in this case. A somewhat hasty generalisation might be to suggest that this is the only counterexample and that uniform estimates hold, when the polynomial is normalised in the $(d-2)$ -nd place, as soon as the degree $d \geq 3$. However, a slightly more subtle counterexample $P(t) = \epsilon t^3 - 2\sqrt{\epsilon}t^2 + t$ exists in this case. Nevertheless, we will see later that uniform estimates do in fact hold when the polynomial is normalised in the $(d-2)$ -nd place provided the polynomial has degree $d \geq 4$.

The above counterexamples generalise to any degree, we simply consider the example $P(t) = \epsilon t^d - 1$, and so we see that uniform estimates will not exist in any degree. Furthermore, the above remarks also lead us to make the following observation. By considering the polynomial

$$P(t) = \epsilon t^{k-1}(t-r)^k$$

where $r = \epsilon^{-1/k}$ with ϵ small, one can observe that there are no uniform estimates for $|S_{\delta,P}|$ in this particular case when $\deg(P) = 2k-1$, and thus we see

that uniform estimates will not hold in general.

Nevertheless, if P is normalised in the $(d - k)$ -th place, then we will see later on that there does indeed exist a $d(k)$, so that whenever $d \geq d(k)$, there exist uniform estimates for $|S_{\delta,P}|$. In fact the example just given above demonstrates that the sharp value of $d(k)$ is $2k$. More precisely, we will prove later on that for each $k \geq 0$ there exist uniform estimates for $|S_{\delta,P}|$ for any normalised real polynomial $P(t) = \sum_{m=0}^d c_m t^m$ with $d \geq 2k$ and $|c_{d-k}| = 1$; that is, there exists an absolute constant $A = A_d$ so that $|S_{\delta,P}| \leq A_d \delta^{1/d}$ whenever $\delta < 1$. We note also that a formulation as above can also be made for global in $\delta > 0$ estimates where the bound in this case is $\delta^{1/(d-k)}$.

We will first establish the above result first for the cases $k = 0$ with $d(0) = 0$, and then we will go on to consider the general k case, for $k \geq 1$, afterwards. We of course already know that the case $k = 0$ is trivially true from the fact that the d -th derivative of P is automatically uniformly bounded from below over the whole real line, and thus we can use Proposition 1.4.1. However, our task is to obtain uniform bounds for $S_{\delta,P}$ by incorporating the finer structure of the sub-level sets themselves, since we have already seen that solely knowing the truth of uniform estimates for global sub-level sets $S_{\delta,P}$ is insufficient to obtain a uniform bound for the sub-level set $\{x \in K : |P(\Phi(x))| < \delta\}$.

In fact, the above claims concerning $|S_{\delta,P}|$, for any normalised real polynomial, actually follow from a much stronger result, which we will prove later on, regarding the *structure* of sub-level sets $S_{\delta,P}$. More precisely, the result is as follows.

Theorem 3.3.7. *For any $k \geq 0$ there exists an absolute constant $A = A_d$ so that for any normalised real polynomial $P(t) = \sum_{m=0}^d c_m t^m$ with $d \geq 2k$ and $|c_{d-k}| = 1$,*

$$\{t \in \mathbf{R} : |P(t)| < \delta\} \subset \bigcup_{\xi \in \mathcal{R}_P} \{t \in \mathbf{R} : |t - \Re(\xi)| \leq A_d \delta^{1/d}\} \quad (3.3.12)$$

whenever $0 \leq \delta < 1$. Here \mathcal{R}_P denotes the set of roots of P .

The case $k = 0$ for the above theorem is now no longer a trivial matter, and we will establish this case with $d(0) = 0$. Again, as we mentioned before in the context of the result for the global sub-level set $S_{\delta,P}$, the above theorem can be refined to give the more precise estimate $\delta^{1/(d-k)}$ which is valid for *all* $\delta > 0$ i.e. globally in $\delta > 0$. As we will see, this theorem can be “bootstrapped” to a theorem about uniform estimates for the multilinear sub-level set operator $S_{\delta}^{P(\Phi),K,\pi}$, given an a priori estimate for the multilinear oscillatory integral operators $\Lambda_{\lambda}^{\Phi,K,\pi}$.

Now, since our main goal is in fact to seek the existence of *global* and *uniform* estimates for the operator $S_{\delta}^{P(\Phi),K,\pi}$, where P is a normalised polynomial of bounded degree d , given that we know an a priori estimate for the corresponding multilinear oscillatory integral $\Lambda_{\lambda}^{\Phi,K,\pi}$; we will now endeavour to formulate more precisely our strategy. We have as a set theoretic identity

$$\{x \in K : |P(\Phi(x))| < \delta\} = K \cap \Phi^{-1}(\mathbf{R} \cap \{t : |P(t)| < \delta\}),$$

and since unions and intersections are preserved under the inverse image of a set, it will therefore be our aim to decompose \mathbf{R} into a finite disjoint union of intervals so that on each interval the polynomial P will look like a centred monomial. In fact, we will prove that the right hand side of the above identity will be contained in a union of sub-level sets where the phase is essentially of a monomial nature. We note that the particular set inclusion, that we wish to establish for the right hand side of the above identity, is, in fact, essentially the substance of the main theorem that we have just formulated a moment ago above.

We will accomplish the above by appealing to a decomposition procedure from S. Dendrinos and J. Wright [12], which is similar to but more refined than a certain decomposition procedure introduced in D. H. Phong and E. M. Stein [32]. In fact, it is pertinent at this point to mention that one can find in the literature a significant body of previous work where decompositions such as the ones given in S. Dendrinos and J. Wright [12], and D. H. Phong and E. M. Stein [32], for example, have similar aims to what we wish to achieve, especially with regards to the case $k = 0$; and for more examples we refer the reader to A. Carbery, F. Ricci and J. Wright [5], S. Dendrinos, M. Folch-Gabayet, and J. Wright [13], M. Folch-Gabayet and J. Wright [16], [17].

For the case $k = 0$, we will see that an application of the decomposition procedure from S. Dendrinos and J. Wright [12] is all that is really required to achieve the sub-level set inclusion result in this particular instance. However, in general, when we consider the case when $k \geq 1$, we will also need in addition to this decomposition procedure some algebra to optimise estimates; and consequently, we will see that the problem of obtaining the uniform sub-level set can be reduced to the problem of showing that an elementary combinatorial inequality holds. More precisely, we will reduce matters to proving that for any $k \geq 1$ and for any collection $\{t_j\}_{j=1}^d \subset \mathbf{C}$ of distinct points with $d \geq 2k$, the following inequality holds

$$1 \lesssim_d |\epsilon| \min_{1 \leq j \leq d} \max_{\ell_1} \max_{\ell_2 \neq \ell_1} \cdots \max_{\ell_k \in \{\ell_1, \ell_2, \dots, \ell_{k-1}\}} \prod_{r=1}^k |t_j - t_{\ell_r}| \quad (3.3.13)$$

when $|\epsilon|, |\epsilon s_1(t_1, \dots, t_d)|, \dots, |\epsilon s_{k-1}(t_1, \dots, t_d)|$ are small, depending only on d , $|\epsilon s_k(t_1, \dots, t_d)| = 1$, and $|\epsilon s_j(t_1, \dots, t_d)| \leq 1$ for every j such that $j \geq k + 1$. Here, for each i such that $1 \leq i \leq d$, the functions s_i are the elementary symmetric polynomials.

The above inequality has the following equivalent formulation. Without loss of generality assume that the minimum in j occurs at $j = 1$ and suppose that $|t_1 - t_2| \leq \cdots \leq |t_1 - t_d|$. Then (3.3.13) can be restated as

$$1 \lesssim_d |\epsilon| |t_1 - t_{d-k+1}| \cdots |t_1 - t_d| \quad (3.3.14)$$

with the same conditions on ϵ and the elementary symmetric functions of t_1, \dots, t_d as before.

Once the sub-level set inclusion is obtained, we will be able to bound the sub-level set operator by a finite sum of simpler sub-level set operators, each having phases essentially of a monomial nature, to which we can then individually apply the folklore procedure. The invariance of oscillatory integral estimates under centred monomial transformations of the phase will be the precise tool responsible for ensuring that the estimate obtained for the simpler sub-level set operators is uniform in each of them.

We will now turn to the proof of the theorem for the case $k = 0$, after which, we will also establish the theorem for the general k case.

3.3.1 Proof of the sub-level set inclusion theorem for the case $k = 0$

Proof. Let \mathcal{R}_P denote the set $\{\xi \in \mathbf{C} : P(\xi) = 0\}$. If $d = 0$, then the above hypothesis implies that $P(t) = 1$ and hence that $\mathcal{R}_P = \emptyset$. Moreover, we obtain, in this particular case that $\{t \in \mathbf{R} : |P(t)| < \delta\} = \{t : 1 < \delta\}$, and consequently we have

$$\{t \in \mathbf{R} : |P(t)| < \delta\} = \begin{cases} \emptyset & \delta \leq 1, \\ \mathbf{R} & \delta > 1. \end{cases}$$

We therefore have the trivial set inclusion

$$\{t \in \mathbf{R} : |P(t)| < \delta\} \subset \{t : |t| < \delta^\infty\},$$

where the symbol δ^∞ is interpreted as 0 if $\delta \leq 1$, and ∞ if $\delta > 1$.

Now let P be a monic polynomial of degree $d \geq 1$. Since some roots may be repeated we have $\#\mathcal{R}_P = d'$ where $d' \leq d$.

Now

$$\mathcal{R}_P = \tilde{\mathcal{R}}_P \cup \mathcal{R}'_P$$

where $\tilde{\mathcal{R}}_P = \{\xi \in \mathcal{R}_P : \Re(\xi) = \xi\}$ and $\mathcal{R}'_P = \{\xi \in \mathcal{R}_P : \Re(\xi) \neq \xi\}$. We factor P with respect to its distinct roots so that

$$P(t) = \prod_{\xi \in \mathcal{R}_P} (t - \xi)^{a_\xi}$$

where a_ξ is the multiplicity of the root ξ so that $\sum_{\xi \in \mathcal{R}_P} a_\xi = d$.

For each $\xi \in \mathcal{R}_P$ we construct the open interval S_ξ defined by

$$\bigcap_{\eta \in \mathcal{R}_P : \eta \neq \xi} \{t \in \mathbf{R} : |t - \xi| < |t - \eta|\}.$$

The real line can then be decomposed as follows

$$\mathbf{R} = \bigcup_{\xi \in \mathcal{R}_P} \bar{S}_\xi.$$

Here \bar{S}_ξ denotes the closure of the set S_ξ in \mathbf{R} with respect to the standard topology. Moreover, it is clear, on account of how the open interval S_ξ is defined, that $S_\xi \cap S_\zeta = \emptyset$ for every ξ and ζ such that $\xi \neq \zeta$.

If P only has real roots, and so $\mathcal{R}'_P = \emptyset$, then for each $\xi \in \mathcal{R}_P$ the interval $S_\xi \neq \emptyset$, and the total number of intervals making up the decomposition of \mathbf{R} is equal to $\#\mathcal{R}_P$.

However, if $\mathcal{R}'_P \neq \emptyset$, then for each $\xi \in \mathcal{R}'_P$ either $S_\xi = \emptyset$ or $S_\xi \neq \emptyset$. Also, since complex roots occur in conjugate pairs we have $\bar{\xi} \in \mathcal{R}'_P$. We observe that $|t - z| = |t - \bar{z}| \quad \forall t \in \mathbf{R} \quad \text{and} \quad \forall z \in \mathbf{C}$, and so we have $S_\xi = S_{\bar{\xi}}$ for each $\xi \in \mathcal{R}'_P$. Hence, since we have in addition the possibility for complex roots ξ that some of the intervals S_ξ , if not all, may be empty; the total number of intervals in the decomposition of \mathbf{R} will be in general much less than $\#\mathcal{R}_P$ when P has complex roots as well.

Now for each $\xi \in \mathcal{R}_P$, whether ξ is a complex root³ or real root, the interval S_ξ can be decomposed further into $O(1)$ disjoint open sets $I_k(\xi)$, where $O(1)$ is controlled by the degree d of the polynomial P , as follows. We label ξ as η_1 and all the other remaining $\xi' \in \mathcal{R}_P \setminus \{\xi\}$ as $\eta_2, \dots, \eta_{d'}$ according to the ordering

$$|\eta_1 - \eta_2| \leq \dots \leq |\eta_1 - \eta_{d'}|,$$

where $\#\mathcal{R}_P = d'$. The product representation of P is now

³If $\xi \in \mathbf{C}$ then either $S_\xi = \emptyset$ or $S_\xi \neq \emptyset$, and if $S_\xi = \emptyset$ then there is nothing to decompose.

$$P(t) = \prod_{j=1}^{d'} (t - \eta_j)^{a_j}$$

and $a_1 + \dots + a_{d'} = d$.

For each $k \geq 1$, we form the open set

$$I_k(\eta_1) = \left\{ t \in S_{\eta_1} : \frac{|\eta_1 - \eta_k|}{2} < |t - \eta_1| < \frac{|\eta_1 - \eta_{k+1}|}{2} \right\},$$

and so

$$\bar{S}_{\eta_1} = \bigcup_{k \geq 1} \bar{I}_k(\eta_1),$$

where \bar{I}_k denotes the closure of I_k ; and clearly $I_k(\eta_1) \cap I_\ell(\eta_1) = \emptyset$ for every k and ℓ such that $k \neq \ell$.

If $\xi = \eta_1$ happens to be a real root, and we also have complex roots as well, say $\eta_\ell \neq \eta_1$ is a complex root for some ℓ , then $\eta_{\ell+1} = \bar{\eta}_\ell$, and so

$$|t - \eta_{\ell+1}| = |t - \eta_\ell| \quad \forall t \in \mathbf{R}$$

and thus $I_{\ell+1}(\eta_1) = \emptyset$. And so in the case of complex roots we will have many $I_k(\eta_1)$ which are possibly empty and thus redundant in the disjoint decomposition of S_{η_1} .

Moreover, even if all the roots are real, we may still encounter situations where certain $I_k(\eta_1)$ are redundant in the disjoint decomposition of S_{η_1} . For instance, if we are in the situation where we have for some $k_0 < d'$ that $\eta_{k_0} < \eta_1 < \eta_2$, and all the other roots not equal to η_{k_0} are greater than η_2 . Then

$$S_{\eta_1} = \left\{ t \in \mathbf{R} : \frac{\eta_1 + \eta_{k_0}}{2} < t < \frac{\eta_1 + \eta_2}{2} \right\},$$

and one can readily see that $I_{k_0}(\eta_1) = \emptyset$, and in fact that $I_k(\eta_1) = \emptyset$ for each k such that $k \geq k_0$; and so in this case we will have

$$S_{\eta_1} = \bigcup_{k=1}^{k_0-1} I_k(\eta_1).$$

Let us just assume for the moment that ξ is a real root. We make two simple observations.

$$t \in I_k(\eta_1) \Rightarrow |t - \eta_1| < \frac{|\eta_1 - \eta_{k+1}|}{2} \leq \frac{|\eta_1 - \eta_j|}{2} \quad \forall j \geq k+1.$$

So that $\forall j \geq k+1$ we have

$$|t - \eta_j| \leq |t - \eta_1| + |\eta_1 - \eta_j| < \frac{3}{2}|\eta_1 - \eta_j|,$$

and

$$|t - \eta_j| \geq |\eta_1 - \eta_j| - |t - \eta_1| > \frac{1}{2}|\eta_1 - \eta_j|,$$

and thus finally

$$|t - \eta_1| \sim |\eta_1 - \eta_j|. \tag{3.3.15}$$

The second observation is

$$t \in I_k(\eta_1) \Rightarrow \frac{|\eta_1 - \eta_j|}{2} < |t - \eta_1| \quad \forall j : 1 \leq j \leq k,$$

and also since $I_k(\eta_1) \subset S_{\eta_1}$ we automatically have $|t - \eta_1| < |t - \eta_j| \quad \forall j \neq 1$.

So for any j such that $1 \leq j \leq k$ we have

$$|t - \eta_1| < |t - \eta_j| < 3|t - \eta_1|,$$

and thus finally

$$|t - \eta_j| \sim |t - \eta_1|. \quad (3.3.16)$$

From the two observations (3.3.15) and (3.3.16) we see that if $t \in I_k(\eta_1)$ then

$$2^{-d}|t - \eta_1|^{A_k} \prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \leq |P(t)| \leq 3^d |t - \eta_1|^{A_k} \prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}$$

where $A_k = a_1 + \dots + a_k$.

Hence, if $t \in I_k(\eta_1)$ then $|P(t)| \sim |t - \eta_1|^{A_k}$ looks like a centred monomial, where the constant in question is equal to $B_d \prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}$ where B_d is either 2^{-d} or 3^d .

Now, if our root ξ is complex, then there are slight adjustments that have to be made regarding the observations we made just a moment ago. We have $\forall t \in \mathbf{R}$ and $\forall \eta_1 \in \mathbf{C}$ that $|t - \eta_1| \geq |t - \Re(\eta_1)|$ and $|t - \eta_1| \geq |\Re(\eta_1) - \eta_1|$. Now depending on where t is on the real line we will either have $|t - \Re(\eta_1)| \geq |\Re(\eta_1) - \eta_1|$ or $|t - \Re(\eta_1)| < |\Re(\eta_1) - \eta_1|$. Hence, we have to split \mathbf{R} as follows

$$\begin{aligned} \mathbf{R} &= \{t \in \mathbf{R} : |t - \Re(\eta_1)| \geq |\Re(\eta_1) - \eta_1|\} \cup \{t \in \mathbf{R} : |t - \Re(\eta_1)| < |\Re(\eta_1) - \eta_1|\} \\ &= T_1 \cup T_2. \end{aligned}$$

It is then easy to see that if $t \in T_1$ then $|t - \eta_1| \sim |t - \Re(\eta_1)|$, and if $t \in T_2$ we have $|t - \eta_1| \sim |\Re(\eta_1) - \eta_1|$. Hence, since

$$I_k(\eta_1) = (I_k(\eta_1) \cap T_1) \cup (I_k(\eta_1) \cap T_2),$$

by making similar observations as before, we obtain that if $t \in I_k(\eta_1)$ then

$$|P(t)| \sim |t - \Re(\eta_1)|^{A_k}$$

looks like a centred monomial, or that

$$|P(t)| \sim |\Re(\eta_1) - \eta_1|^{A_k}$$

looks like a constant. The constant is again equal to $B_d \prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}$.

So having got this far, our next main task will be to show that, in general, for every root ξ of P , whether it be real or complex, and every $I_k(\xi) \subset S_\xi$, we have the set inclusion

$$I_k(\xi) \cap \{t \in \mathbf{R} : |P(t)| < \delta\} \subset \{t \in \mathbf{R} : |t - \Re(\xi)| \leq 2\delta^{1/d}\}. \quad (3.3.17)$$

The main corpus of our work will then be complete. For then (3.3.17) and the fact that we have

$$\mathbf{R} = \bigcup_{\xi \in \mathcal{R}_P} \bigcup_{k \geq 1} I_k(\xi),$$

allows us to write

$$\begin{aligned} \mathbf{R} \cap \{t : |P(t)| < \delta\} &= \bigcup_{\xi \in \mathcal{R}_P} \bigcup_{k \geq 1} (I_k(\xi) \cap \{t \in \mathbf{R} : |P(t)| < \delta\}) \\ &\subset \bigcup_{\xi \in \mathcal{R}_P} \{t \in \mathbf{R} : |t - \Re(\xi)| \leq 2\delta^{1/d}\}, \end{aligned}$$

thus completing the proof.

So let us now prove that for every root ξ of P , whether it be real or complex, and every $I_k(\xi) \subset S_\xi$, that we have the set inclusion given in (3.3.17).

Since we have in general that

$$I_k(\eta_1) = (I_k(\eta_1) \cap T_1) \cup (I_k(\eta_1) \cap T_2),$$

the most general case to consider is when $I_k(\eta_1) \cap T_i \neq \emptyset \quad \forall i = 1, 2$.

Observe that $\forall i = 1, 2$ we have

$$I_k(\eta_1) \cap T_i \cap \{t : |P(t)| < \delta\} \subset \{t \in \mathbf{R} : |t - \Re(\eta_1)| \leq \ell(\delta, A_k)\}, \quad (3.3.18)$$

where

$$\ell(\delta, A_k) = \min \left(\frac{|\eta_1 - \eta_{k+1}|}{2}, \left(\frac{2^d \delta}{\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}} \right)^{\frac{1}{A_k}} \right).$$

It now remains for us to show that $\ell(\delta, A_k) \leq 2\delta^{1/d}$, and we do this by considering the two cases depending on whether

$$|\eta_1 - \eta_{k+1}|^{A_k} \leq 2^d 2^{A_k} \delta \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \right)^{-1}$$

or

$$|\eta_1 - \eta_{k+1}|^{A_k} \geq 2^d 2^{A_k} \delta \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \right)^{-1}.$$

Note. We recall that $A_k = a_1 + \dots + a_k$ and so $d = A_k + a_{k+1} + \dots + a_{d'}$.

$$\text{Case (1) } |\eta_1 - \eta_{k+1}|^{A_k} \leq 2^d 2^{A_k} \delta \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \right)^{-1}.$$

The above condition implies that

$$\begin{aligned}
\ell(\delta, A_k) &= \frac{|\eta_1 - \eta_{k+1}|}{2} \\
&= \frac{1}{2} (|\eta_1 - \eta_{k+1}|^{A_k} |\eta_1 - \eta_{k+1}|^{a_{k+1} + \dots + a_{d'}})^{1/d} \\
&\leq \frac{1}{2} \left(2^d 2^{A_k} \delta \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \right)^{-1} |\eta_1 - \eta_{k+1}|^{a_{k+1} + \dots + a_{d'}} \right)^{1/d}.
\end{aligned}$$

Finally, the prior ordering of the roots so that

$$|\eta_1 - \eta_2| \leq \dots \leq |\eta_1 - \eta_{k+1}| \leq |\eta_1 - \eta_{k+2}| \leq \dots \leq |\eta_1 - \eta_{d'}|$$

implies that

$$|\eta_1 - \eta_{k+1}|^{a_{k+1} + \dots + a_{d'}} \leq \prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}. \quad (3.3.19)$$

Thus giving us the conclusion that

$$\ell(\delta, A_k) \leq 2\delta^{1/d}.$$

$$\text{Case (2) } |\eta_1 - \eta_{k+1}|^{A_k} \geq 2^d 2^{A_k} \delta \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \right)^{-1}.$$

The above condition implies that

$$\begin{aligned}
\ell(\delta, A_k) &= \left(\frac{2^d \delta}{\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}} \right)^{\frac{1}{A_k}} \\
&= \left(\frac{2^{d/A_k} \delta^{\left(\frac{1}{A_k} - \frac{1}{d}\right)}}{\left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j} \right)^{\frac{1}{A_k}}} \right) \delta^{1/d}.
\end{aligned}$$

Now, by applying the condition in our assumption above once again, and also using the fact that the roots satisfy (3.3.19), it follows that

$$\begin{aligned}
\frac{2^{d/A_k} \delta^{\left(\frac{1}{A_k} - \frac{1}{d}\right)}}{\left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}\right)^{\frac{1}{A_k}}} &\leq \frac{2^{d/A_k} 2^{-(d+A_k)\left(\frac{1}{A_k} - \frac{1}{d}\right)} \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}\right)^{\left(\frac{1}{A_k} - \frac{1}{d}\right)} |\eta_1 - \eta_{k+1}|^{1 - \frac{A_k}{d}}}{\left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}\right)^{\frac{1}{A_k}}} \\
&= 2^{A_k/d} \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}\right)^{-1/d} |\eta_1 - \eta_{k+1}|^{\frac{a_{k+1} + \dots + a_{d'}}{d}} \\
&\leq 2^{A_k/d} \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}\right)^{-1/d} \left(\prod_{j=k+1}^{d'} |\eta_1 - \eta_j|^{a_j}\right)^{1/d} \\
&= 2^{A_k/d} \\
&= 2^1 2^{-\frac{a_{k+1} + \dots + a_{d'}}{d}} \leq 2.
\end{aligned}$$

Thus giving us the conclusion that

$$\ell(\delta, A_k) \leq 2\delta^{1/d}.$$

Hence, we can conclude that for every root ξ of P and every $I_k(\xi) \subset S_\xi$, we have the set inclusion

$$I_k(\xi) \cap \{t \in \mathbf{R} : |P(t)| < \delta\} \subset \{t \in \mathbf{R} : |t - \Re(\xi)| \leq 2\delta^{1/d}\}.$$

□

Remark. Before we go on to prove the general case of the theorem i.e. the case where we have $|c_{d-k}| = 1$ for $k \geq 1$, that is, the case where the normalisation occurs in the k -th coefficient of P , it is pertinent at this point to make an important observation about the proof for the case $k = 0$ that we have just given; as it will prove useful to us later on when addressing the general k case. If we go

back carefully over the proof we have just given, we can see that it actually gives a much stronger result. For, not only is it *global* in $\delta > 0$, but the polynomial P does not have to be monic.

Observe that if we had $P(t) = \epsilon t^d + \dots$ i.e. $P(t) = \epsilon \prod_{\xi \in \mathcal{R}_P} (t - \xi)^{a_\xi}$, with $\epsilon \neq 1$, then the proof still works if $\nu_d \leq |\epsilon| < 1$, or if $1 < \sigma_d \leq |\epsilon|$, where ν_d and σ_d are constants only depending on the degree d . The only change that occurs is in the magnitude of the absolute constant A_d that appears in the uniform estimate $A_d \delta^{1/d}$. In the first instance, the absolute constant A_d that we obtain in the estimate $A_d \delta^{1/d}$ increases since $1 < \frac{1}{|\epsilon|} \leq \frac{1}{\nu_d}$, and in the second instance the absolute constant A_d that we obtain in the estimate $A_d \delta^{1/d}$ decreases since $\frac{1}{|\epsilon|} \leq \frac{1}{\sigma_d} < 1$.

3.4 The general k case

For any polynomial P of degree d we have that

$$P(t) = C \prod_{j=1}^d (t - r_j),$$

where the roots r_j may be repeated. Moreover, it is well known that if we expand the above product then we obtain the following expression for P

$$P(t) = C s_0 t^d - C s_1 t^{d-1} + C s_2 t^{d-2} + \dots + C(-1)^{d-1} s_{d-1} t + C(-1)^d s_d \quad (3.4.1)$$

where each $s_n = s_n(r_1, \dots, r_d)$ is a polynomial in r_1, \dots, r_d and is given by

$$s_n(r_1, \dots, r_d) = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq d} r_{i_1} \cdots r_{i_n}.$$

So for instance, when $n = 0$ there is only the empty product to sum over and as a

result $s_0(r_1, \dots, r_d) = 1$, $s_1(r_1, \dots, r_d) = r_1 + \dots + r_d$ and $s_d(r_1, \dots, r_d) = r_1 \cdots r_d$, while for $n > d$ no products at all can be formed, so $s_n(r_1, \dots, r_d) = 0$ in these cases. The polynomials s_1, \dots, s_d are symmetric, and very special indeed, since it can be shown that any general symmetric polynomial can be expressed as a polynomial of these basic symmetric polynomials, and as a result of this fact the polynomials s_1, \dots, s_d are called the *elementary symmetric polynomials*.

This fundamental fact of symmetric polynomials will prove to be very useful for us, and we will state the result formally without proof. We refer the interested reader to [26] should they wish to consult the proof; but before we go on to state the fundamental result of symmetric polynomials, we will first, for the reader's convenience, and also because it plays an important role in our future analysis, introduce a basic notion that will be utilised in the particular formulation which we will require. Let X_1, \dots, X_n be variables. We define the *weight* of a monomial

$$X_1^{\nu_1} \cdots X_n^{\nu_n}$$

to be $\nu_1 + 2\nu_2 + \dots + n\nu_n$. We define the *weighted degree* of a polynomial $g(X_1, \dots, X_n)$ to be the maximum of the weights of the monomials occurring in g .

Theorem 3.4.1. [26] *If $f \in \mathbf{R}[X_1, \dots, X_n]$ is a symmetric polynomial of degree L , then there exists a unique polynomial $g \in \mathbf{R}[X_1, \dots, X_n]$ of weighted degree $\leq L$ such that*

$$f(X_1, \dots, X_n) = g(s_1(X_1, \dots, X_n), \dots, s_n(X_1, \dots, X_n)).$$

Note. g being of *weighted degree* $\leq L$ means that

$$g(X_1, \dots, X_n) = \sum_{\alpha: \sum_{j=1}^n j\alpha_j \leq L} b_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index.

Remark. The polynomial g actually has degree at most L in the usual sense, and many applications of Theorem 3.4.1 only use this fact. However, it is an interesting feature of our analysis that we will utilise the full strength of the above result, namely that g has in fact *weighted degree* at most L .

Having recalled the well known algebraic fact that the coefficients of any polynomial P can be described in terms of the elementary symmetric polynomials of its roots; we will see that the problem, of proving the uniform set inclusion in the general k case, will be narrowed down to the task of proving an elementary combinatorial inequality. This basic inequality will then yield the uniform result that we are seeking. We will prove this combinatorial inequality by constructing a suitable symmetric polynomial Q which we will then be able to bound from below by a constant depending only on the degree of P .

We will need to find a suitable way of relating the normalisation hypothesis on the coefficients of P to the task of proving that the symmetric polynomial Q is bounded below, and, since equation (3.4.1) tells us that the coefficients of any polynomial are given in terms of symmetric polynomials of the roots, we will see that Theorem 3.4.1 will provide us with the appropriate means of achieving this aim. Nevertheless, before we do all this, we first need to introduce some notation.

Notation: For integers m and n such that $m \leq n$, let $[m, n]$ denote the set of integers $\{m, m+1, \dots, n\}$.

Let $d \geq 2k$, we will now consider the following symmetric polynomial in t_1, \dots, t_d

$$Q(t_1, \dots, t_d) = \sum_{j_1 < \dots < j_{2k}} \sum_{\sigma \in \mathcal{S}_{2k}} \prod_{\ell=1}^k (t_{j_{\sigma(2\ell-1)}} - t_{j_{\sigma(2\ell)}})^2$$

where \mathcal{S}_{2k} denotes the symmetric group of permutations on the set $\{1, 2, \dots, 2k\}$ and the first sum is taken over all $2k$ -tuples (j_1, \dots, j_{2k}) of increasing elements in $[1, d]$.

Note. Q is a symmetric polynomial that is homogeneous of degree $2k$.

This particular symmetric polynomial has some very nice properties which will prove to be very useful in our efforts towards establishing the general k case uniform set inclusion. In particular it has the very nice property that

$$Q(t_1, \dots, t_k, 0, \dots, 0) = c[t_1 \cdots t_k]^2$$

for some $c > 0$. Moreover, the most notable property of Q is that it satisfies the following crucial inequality.

Lemma 3.4.2. *Let $d \geq 2k$. Suppose the collection $\{t_j\}_{j=1}^d \subset \mathbf{C}$ of distinct points satisfies the ordering*

$$|t_1 - t_2| \leq |t_1 - t_3| \leq \dots \leq |t_1 - t_d|,$$

then the symmetric polynomial

$$Q(t_1, \dots, t_d) = \sum_{j_1 < \dots < j_{2k}} \sum_{\sigma \in \mathcal{S}_{2k}} \prod_{\ell=1}^k (t_{j_{\sigma(2\ell-1)}} - t_{j_{\sigma(2\ell)}})^2$$

obeys the estimate

$$|Q(t_1, \dots, t_d)| \lesssim_d |t_1 - t_{d-k+1}|^2 \cdots |t_1 - t_d|^2.$$

Proof. Fix a tuple (j_1, \dots, j_{2k}) and a permutation $\sigma \in \mathcal{S}_{2k}$, and consider the corresponding summand

$$\prod_{\ell=1}^k (t_{j_{\sigma(2\ell-1)}} - t_{j_{\sigma(2\ell)}})^2.$$

We claim that there exists an ℓ_0 , with $1 \leq \ell_0 \leq k$, such that

$$\max(j_{\sigma(2\ell_0-1)}, j_{\sigma(2\ell_0)}) \leq d - k + 1.$$

Otherwise, for every ℓ such that $1 \leq \ell \leq k$ we have

$$\max(j_{\sigma(2\ell-1)}, j_{\sigma(2\ell)}) \geq d - k + 2,$$

and then, by setting $j_{\sigma_\ell} = \max(j_{\sigma(2\ell-1)}, j_{\sigma(2\ell)})$, we can find k distinct integers such that $\{j_{\sigma_1}, \dots, j_{\sigma_k}\} \subset [d - k + 2, d]$, which is clearly impossible since $\#[d - k + 2, d] = k - 1$.

Hence, there exists a pair $(\sigma(2\ell_0 - 1), \sigma(2\ell_0))$, such that

$$\max(j_{\sigma(2\ell_0-1)}, j_{\sigma(2\ell_0)}) \leq d - k + 1,$$

with ℓ_0 satisfying $1 \leq \ell_0 \leq k$. Now, for such a pair $(\sigma(2\ell_0 - 1), \sigma(2\ell_0))$ we observe that

$$|t_{j_{\sigma(2\ell_0-1)}} - t_{j_{\sigma(2\ell_0)}}| \leq 2|t_1 - t_{d-k+1}|.$$

We eliminate this pair and proceed as before. By following the same argument given above, one can see that there must exist $\ell_1 \in [1, k] \setminus \{\ell_0\}$ such that

$$\max(j_{\sigma(2\ell_1-1)}, j_{\sigma(2\ell_1)}) \leq d - k + 2.$$

One then sees that

$$|t_{j_{\sigma(2\ell_1-1)}} - t_{j_{\sigma(2\ell_1)}}| \leq 2|t_1 - t_{d-k+2}|.$$

We eliminate the pair $(\sigma(2\ell_1 - 1), \sigma(2\ell_1))$, and continue in this inductive manner until the process terminates.

Note. Since there are k pairs to consider from the outset, this process will terminate after at most k steps. If we are in the situation where we only eliminate one pair at each step, then at the k -th step there will be only one $\ell_{k-1} \in [1, k] \setminus \bigcup_{n=0}^{k-2} \{\ell_n\}$. For the final pair $(\sigma(2\ell_{k-1} - 1), \sigma(2\ell_{k-1}))$ we must have

$$\max(j_{\sigma(2\ell_{k-1}-1)}, j_{\sigma(2\ell_{k-1})}) \leq d - k + k = d.$$

Otherwise, by setting $j_{\sigma(2\ell_{k-1})} = \max(j_{\sigma(2\ell_{k-1}-1)}, j_{\sigma(2\ell_{k-1})})$, we will have

$$\{j_{\sigma(2\ell_{k-1})}\} \subset [d - k + k + 1, d] = [d + 1, d] = \emptyset,$$

which is clearly impossible. Hence, for the final pair we observe that

$$|t_{j_{\sigma(2\ell_{k-1}-1)}} - t_{j_{\sigma(2\ell_{k-1})}}| \leq 2|t_1 - t_d|.$$

□

We are now in a suitable position to give our full attention to the proof of the general k case.

3.4.1 Proof of Theorem 3.3.7

Before we embark upon the proof of Theorem 3.3.7, it is pertinent at this point to recall a previous comment, which was made briefly in the paragraphs running up to the statement of Theorem 3.3.7, about the sharpness of the restriction $d \geq 2k$.

The restriction is sharp in the following sense. If a polynomial P has degree $d < 2k$ then the uniform set inclusion (3.3.12) will fail to be true. This can be seen by considering the polynomial $P(t) = \epsilon t^{k-1}(t - r)^k$, where $r = \epsilon^{-1/k}$ and ϵ is small. This polynomial has degree $2k - 1$, and one can observe that there are no uniform estimates for $|S_{\delta, P}|$ in this case. This in turn then implies that it is impossible for the uniform set inclusion (3.3.12) to hold, for if such a uniform set inclusion were also true, then we could also at the same time obtain uniform estimates for $|S_{\delta, P}|$, thus giving a contradiction.

Proof. It is convenient to prove a slight generalisation of Theorem 3.3.7; namely, for any $\sigma > 0$, we will establish the set inclusion (3.3.12) for all polynomials $P(t) = \sum_{j=0}^d c_j t^j$ satisfying the relaxed normalisation conditions $|c_j| \leq 1$ for all $j \geq 0$ and $|c_{d-k}| \geq \sigma$ (the constant A_d will now depend on σ as well).

We will prove this by induction on k . The case $k = 0$ has been established already (we refer the reader to the *Remark* after the proof of the $k = 0$ case). The induction will be carried out by establishing the following stronger result. For each $k \geq 0$ and every $\sigma > 0$, there are small positive constants $\sigma_0, \dots, \sigma_{k-1}$, depending on σ and $d (\geq 2k)$ so that for any polynomial $Q(t) = b_d t^d + \dots + b_1 t + b_0$ with $\sigma \leq |b_{d-k}| \leq 1$, $|b_{d-j}| \leq \sigma_j$ for each j such that $0 \leq j \leq k-1$, and $|b_{d-j}| \leq 1$ for each j such that $j \geq k+1$,

$$\{t \in \mathbf{R} : |Q(t)| < \delta\} \subset \bigcup_{\xi \in \mathcal{R}_Q} \{t \in \mathbf{R} : |t - \Re(\xi)| \leq A\delta^{1/(d-k)}\} \quad (3.4.2)$$

holds for *all* $\delta > 0$, where A depends only on σ , d , and the σ_j 's.

The case $k = 0$ coincides with the desired generalisation of Theorem 3.3.7 with the added bonus that the set inclusion holds for *all* $\delta > 0$ (again the *Remark* after the proof of $k = 0$ case settles this case). However, we will *not* proceed by

induction to establish this stronger result for general k , instead we will do this directly for every k .

Before proceeding to the proof of this stronger result (3.4.2), let us see how this implies the slight generalisation of Theorem 3.3.7 (and hence Theorem 3.3.7 itself) formulated above. Recall that we have proved the case $k = 0$ and we proceed to the induction step, assuming the desired conclusion holds for all values $k' < k$. Let $\sigma > 0$ be given and fix a polynomial $P(t) = \sum_{j=0}^d c_j t^j$ satisfying the relaxed normalisation conditions $|c_j| \leq 1$ for each j such that $j \geq 0$, and $|c_{d-k}| \geq \sigma$.

For this k and $\sigma > 0$, the stronger result produces small positive constants $\sigma_0, \dots, \sigma_{k-1}$ so that (3.4.2) holds for any polynomial $Q(t) = \sum_{j=0}^d b_j t^j$ satisfying the relaxed normalisation conditions, together with the added conditions $|b_{d-j}| \leq \sigma_j$ for each j such that $0 \leq j \leq k-1$.

Returning to our polynomial P , we see that if there is some coefficient c_{d-j} , with $0 \leq j \leq k-1$, satisfying $|c_{d-j}| \geq \sigma_j$, then we can apply the induction hypothesis with $k' = j < k$ to conclude that (3.3.12) holds. On the other hand, if all the coefficients c_{d-j} , with $0 \leq j \leq k-1$, satisfy $|c_{d-j}| \leq \sigma_j$, then (3.4.2) holds and this implies (3.3.12) holds as well when $\delta > 0$ is restricted to be smaller than 1. This completes the induction step.

We now turn to establish (3.4.2). Without loss of generality it suffices to consider the case when all the roots of P are distinct with multiplicity equal to 1. For suppose we are in the situation where one of the roots ξ , say, is repeated with multiplicity a_ξ ; then we can perturb its real part by a_ξ small distances $\nu_1, \dots, \nu_{a_\xi-1}, \nu_{a_\xi}$ so that, setting $\tilde{\xi} = \xi - \Re(\xi)$ and $\tilde{\nu}_i = \Re(\xi) + \nu_i$, we obtain the distinct set of points $\tilde{\xi} + \tilde{\nu}_1, \dots, \tilde{\xi} + \tilde{\nu}_{a_\xi-1}, \tilde{\xi} + \tilde{\nu}_{a_\xi}$ in \mathbf{C} , and thus, together with

the other remaining roots, we are now in the situation where we have distinct points in \mathbf{C} . We can then carry out the analysis of this case and let $\nu_j \rightarrow 0$ for each $j = 1, \dots, a_\xi - 1, a_\xi$ afterwards.

So let us assume then that we have a polynomial P of degree $d \geq 2k$ satisfying the above hypotheses and having d distinct roots, with multiplicity 1, so that $\mathcal{R}_P = \{r_1, \dots, r_d\}$. Hence, we can factor P as according to its distinct roots, so that

$$P(t) = \sum_{j=0}^d c_j t^j = \epsilon \prod_{i=1}^d (t - r_i),$$

where ϵ of course denotes the coefficient c_d . We will recall many ideas from the proof of the $k = 0$ case, the most notable one of course being the decomposition procedure, and so, we will omit certain steps, wherever it is possible to do so, in order to avoid unnecessary repetition.

The only place where novel work is carried out is when we have to prove that certain constants are bounded above by an absolute constant only depending on the degree d ; and this in turn will be achieved once we establish a certain fundamental combinatorial inequality. Therefore, we advise the reader to re-visit the proof of the $k = 0$ case, should they wish to recall certain steps, and also the genesis of certain considerations, which will be stated here without their immediate motivation.

So, in light of our comments at the beginning of the proof, we may suppose that, in terms of the elementary symmetric polynomials of the roots of P , the constants

$$|\epsilon|, |\epsilon s_1(r_1, \dots, r_d)|, \dots, |\epsilon s_{k-1}(r_1, \dots, r_d)|$$

are small, depending only on the degree d , $|\epsilon s_k(r_1, \dots, r_d)|$ is about 1, and $|\epsilon s_j(r_1, \dots, r_d)| \leq 1$ for every j such that $j \geq k+1$.

Note. We will determine how small we need the constants

$$|\epsilon|, |\epsilon s_1(r_1, \dots, r_d)|, \dots, |\epsilon s_{k-1}(r_1, \dots, r_d)|$$

to be towards the end of the proof.

Starting with the polynomial P factored according to its roots so that

$$P(t) = \sum_{j=0}^d c_j t^j = \epsilon \prod_{i=1}^d (t - r_i),$$

we consider each root in turn. For each root r_i we define the intervals S_{r_i} to be

$$\bigcap_{\substack{j=1 \\ j \neq i}}^d \{t \in \mathbf{R} : |t - r_i| < |t - r_j|\},$$

and carry out the decomposition procedure in exactly the same way as before. We first decompose \mathbf{R} according to the intervals S_{r_i} , and then for each root r_i we decompose each S_{r_i} further into $O(1)$ disjoint open sets $I_k(r_i)$, where $O(1)$ is controlled by the degree d of the polynomial P , as follows. We label r_i as η_1 and all the other remaining roots in $\mathcal{R}_P \setminus \{r_i\}$ as η_2, \dots, η_d according to the ordering

$$|\eta_1 - \eta_2| \leq |\eta_1 - \eta_3| \leq \dots \leq |\eta_1 - \eta_d|.$$

For each $m \geq 1$, we form the open set

$$I_m(\eta_1) = \left\{ t \in S_{\eta_1} : \frac{|\eta_1 - \eta_m|}{2} < |t - \eta_1| < \frac{|\eta_1 - \eta_{m+1}|}{2} \right\},$$

and so

$$\bar{S}_{\eta_1} = \bigcup_{m \geq 1} \bar{I}_m(\eta_1),$$

where \bar{I}_m denotes the closure of I_m ; and clearly $I_m(\eta_1) \cap I_\ell(\eta_1) = \emptyset$ for every m and ℓ such that $m \neq \ell$.

The only novel work that we have to carry out in this proof is to show that for every root r_i of P , whether it be real or complex, and every $I_m(r_i) \subset S_{r_i}$, we have the set inclusion

$$I_m(r_i) \cap \{t \in \mathbf{R} : |P(t)| < \delta\} \subset \{t \in \mathbf{R} : |t - \Re(r_i)| \lesssim_d \delta^{1/(d-k)}\}, \quad (3.4.3)$$

where the subscript d in the inequality sign above indicates that the absolute constant, which has been subsumed via this notation, only depends on $\deg(P) = d$. Then the inclusion with $\delta^{1/d}$ instead of $\delta^{1/(d-k)}$ follows since $0 < \delta < 1$.

Now, for a fixed root r_i labeled as η_1 , and all the other remaining roots labeled as before, that is, according to the special ordering

$$|\eta_1 - \eta_2| \leq |\eta_1 - \eta_3| \leq \dots \leq |\eta_1 - \eta_d|,$$

the above set inclusion (3.4.3) is accomplished by showing $\ell(\delta, m) \lesssim_d \delta^{1/(d-k)}$, where

$$\ell(\delta, m) = \min \left(\frac{|\eta_1 - \eta_{m+1}|}{2}, \left(\frac{2^d \delta}{|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|} \right)^{\frac{1}{m}} \right).$$

If the reader wishes to recall how $\ell(\delta, m)$ arises, then we encourage the reader to consult (3.3.18) in the part of the proof of the $k = 0$ case that deals with proving the set inclusion

$$I_m(\xi) \cap \{t \in \mathbf{R} : |P(t)| < \delta\} \subset \{t \in \mathbf{R} : |t - \Re(\xi)| \lesssim_d \delta^{1/d}\},$$

where in this particular context ξ denotes an arbitrary root of P having multi-

plicity a_ξ .

Note. In the proof of the $k = 0$ case we actually have $\ell(\delta, A_m)$, where the number A_m is defined as $A_m = a_1 + \dots + a_m$, with a_1, \dots, a_m being the multiplicities of the various roots. Thus, in particular, $d = A_m + a_{m+1} + \dots + a_{d'}$, where of course $d' \leq d$. Now, since we are considering the case where a_j is equal to 1 for each root, we thus have $A_m = m$.

We prove $\ell(\delta, m) \lesssim_d \delta^{1/(d-k)}$ by considering the two cases depending on whether

$$|\eta_1 - \eta_{m+1}|^m \leq 2^d 2^m \delta \left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j| \right)^{-1}$$

or

$$|\eta_1 - \eta_{m+1}|^m \geq 2^d 2^m \delta \left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j| \right)^{-1}.$$

For the sake of brevity, we will only treat the second case, as the first case is treated in an almost identical way to the manner in which it is dealt with in the proof of the $k = 0$ case. The second condition implies that

$$\begin{aligned} \ell(\delta, m) &= \left(\frac{2^d \delta}{|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|} \right)^{\frac{1}{m}} \\ &= \left(\frac{2^{d/m} \delta^{\left(\frac{1}{m} - \frac{1}{d-k}\right)}}{\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j| \right)^{\frac{1}{m}}} \right) \delta^{1/(d-k)}. \end{aligned}$$

Hence, if we can show that

$$\frac{2^{d/m} \delta^{\left(\frac{1}{m} - \frac{1}{d-k}\right)}}{\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|\right)^{\frac{1}{m}}} \lesssim_d 1, \quad (3.4.4)$$

then we are done. Now, we have by hypothesis that

$$\delta \leq |\eta_1 - \eta_{m+1}|^m \left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j| \right).$$

Note. We have dropped the constant $2^d 2^m$ for the sake of convenience.

So, we write

$$\begin{aligned} \frac{\delta^{\left(\frac{1}{m} - \frac{1}{d-k}\right)}}{\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|\right)^{\frac{1}{m}}} &\leq \frac{\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|\right)^{\left(\frac{1}{m} - \frac{1}{d-k}\right)} |\eta_1 - \eta_{m+1}|^{1 - \frac{m}{d-k}}}{\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|\right)^{\frac{1}{m}}} \\ &= |\epsilon|^{\frac{-1}{d-k}} \left(\prod_{j=m+1}^d |\eta_1 - \eta_j| \right)^{\frac{-1}{d-k}} |\eta_1 - \eta_{m+1}|^{\frac{d-k-m}{d-k}} \\ &= \left(\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j| \right)^{-1} |\eta_1 - \eta_{m+1}|^{d-k-m} \right)^{\frac{1}{d-k}}. \end{aligned}$$

Now, since we know the roots are ordered so that

$$|\eta_1 - \eta_{m+1}|^{d-k-m} \leq |\eta_1 - \eta_{m+1}| \cdots |\eta_1 - \eta_{d-k}|,$$

we then see that

$$\begin{aligned}
\left(\frac{|\eta_1 - \eta_{m+1}|^{d-k-m}}{|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j|} \right)^{\frac{1}{d-k}} &\leq \left(\left(|\epsilon| \prod_{j=m+1}^d |\eta_1 - \eta_j| \right)^{-1} |\eta_1 - \eta_{m+1}| \cdots |\eta_1 - \eta_{d-k}| \right)^{\frac{1}{d-k}} \\
&= \left(\left(|\epsilon| \prod_{j=d-k+1}^d |\eta_1 - \eta_j| \prod_{j=m+1}^{d-k} |\eta_1 - \eta_j| \right)^{-1} \prod_{j=m+1}^{d-k} |\eta_1 - \eta_j| \right)^{\frac{1}{d-k}} \\
&= (|\epsilon| |\eta_1 - \eta_d| \cdots |\eta_1 - \eta_{d-k+1}|)^{\frac{-1}{d-k}}.
\end{aligned}$$

Therefore, in order to prove that (3.4.4) holds, it only remains for us to establish the combinatorial inequality

$$1 \lesssim_d |\epsilon| |\eta_1 - \eta_{d-k+1}| \cdots |\eta_1 - \eta_d|.$$

This is precisely the moment where we can appeal to Lemma 3.4.2, from which we immediately obtain the inequality

$$|Q(\eta_1, \dots, \eta_d)| \lesssim_d |\eta_1 - \eta_{d-k+1}|^2 \cdots |\eta_1 - \eta_d|^2.$$

All that remains for us to do now is to bound $|Q(\eta_1, \dots, \eta_d)|$ from below by a constant depending only on d .

Recall that the collection of points $\{\eta_\ell\}_{\ell=1}^d$ are obtained from the collection $\{r_\ell\}_{\ell=1}^d$ by first labeling a particular fixed root in this collection as η_1 , and then labeling the remaining ones η_2, \dots, η_d so that the ordering $|\eta_1 - \eta_2| \leq \dots \leq |\eta_1 - \eta_d|$ holds for the collection $\{\eta_\ell\}_{\ell=1}^d$. Hence, we can essentially think of $\{\eta_\ell\}_{\ell=1}^d$ as being the result of a particular permutation of $\{r_\ell\}_{\ell=1}^d$. We can therefore write $\eta_\ell = r_{\tau(\ell)}$ for each ℓ such that $1 \leq \ell \leq d$, where τ is a particular permutation that permutes the collection of points $\{r_\ell\}_{\ell=1}^d$ accordingly so that, for a particular fixed root which is mapped to η_1 , they form the resulting collection of points $\{\eta_\ell\}_{\ell=1}^d$ that

satisfies the ordering $|\eta_1 - \eta_2| \leq \dots \leq |\eta_1 - \eta_d|$.

Now Q is a symmetric polynomial, and so it remains invariant under any permutation of its variables. Hence, it follows that $Q(r_{\tau(1)}, \dots, r_{\tau(d)}) = Q(r_1, \dots, r_d)$, and as a consequence, we therefore obtain

$$Q(\eta_1, \dots, \eta_d) = Q(r_{\tau(1)}, \dots, r_{\tau(d)}) = Q(r_1, \dots, r_d),$$

which in turn finally leads to the conclusion

$$|Q(r_1, \dots, r_d)| \lesssim_d |\eta_1 - \eta_{d-k+1}|^2 \dots |\eta_1 - \eta_d|^2.$$

So, to complete the proof, we must now show that $|Q(r_1, \dots, r_d)|$ is bounded from below by a constant dependent at most upon the degree of P . To accomplish this, we must now put ourselves in the situation where we can utilise the hypotheses that the constants

$$|\epsilon|, |\epsilon s_1(r_1, \dots, r_d)|, \dots, |\epsilon s_{k-1}(r_1, \dots, r_d)|$$

are small, depending only on d , and that $|\epsilon s_j(r_1, \dots, r_d)| \leq 1$ for every j such that $j \geq k+1$, with $|\epsilon s_k(r_1, \dots, r_d)| \sim 1$.

We now opt to consider the symmetric polynomial Q as a function of d variables. Since $Q(t_1, \dots, t_d)$ is a symmetric polynomial of degree $2k$, we can thus put Theorem 3.4.1 to work. This theorem guarantees us the existence of a *unique* polynomial $T(x_1, \dots, x_d)$ of weighted degree $\leq 2k$ such that

$$Q(t_1, \dots, t_d) = T(s_1(t_1, \dots, t_d), \dots, s_d(t_1, \dots, t_d)),$$

where s_j are the elementary symmetric polynomials.

Note. Setting $s_n = s_n(t_1, \dots, t_d)$ for each n such that $1 \leq n \leq d$, we can write

$$T(s_1, \dots, s_d) = \sum_{\alpha: \sum_{j=1}^d j\alpha_j \leq 2k} c_\alpha s_1^{\alpha_1} \cdots s_d^{\alpha_d}, \quad (3.4.5)$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index.

Now, since Q is a homogeneous polynomial of degree $2k$ and since each monomial $s_1^{\alpha_1} \cdots s_d^{\alpha_d}$ in (3.4.5) is homogeneous polynomial of degree $\alpha_1 + 2\alpha_2 + \dots + d\alpha_d$ in t_1, \dots, t_d , we observe that

$$T(s_1, \dots, s_d) = \sum_{\alpha: \sum_{j=1}^d j\alpha_j = 2k} c_\alpha s_1^{\alpha_1} \cdots s_d^{\alpha_d}.$$

Furthermore, since $d \geq 2k$, any tuple $(\alpha_1, \dots, \alpha_d)$ that satisfies the constraint condition $\sum_{j=1}^d j\alpha_j = 2k$ must have $\alpha_{2k+1} = \dots = \alpha_d = 0$, otherwise the constraint condition will be violated. Hence we have in fact that

$$T(s_1, \dots, s_d) = \sum_{\alpha: \sum_{j=1}^{2k} j\alpha_j = 2k} c_\alpha s_1^{\alpha_1} \cdots s_{2k}^{\alpha_{2k}}. \quad (3.4.6)$$

Notation: Let Δ denote the set of tuples $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $\sum_{j=1}^{2k} j\alpha_j = 2k$. It is of course obvious that every tuple $(\alpha_1, \dots, \alpha_d)$ in Δ has $\alpha_{2k+1} = \dots = \alpha_d = 0$.

Now, going back to the original expression for Q for a moment, observe that if we leave t_1, \dots, t_k alone and set $t_{k+1} = \dots = t_d = 0$ in it, then

$$Q(t_1, \dots, t_k, 0, \dots, 0) = c[t_1 \cdots t_k]^2$$

for some $c > 0$.

Note. The product $t_1 \cdots t_k$ is in fact the k -th elementary symmetric polynomial

of the variables t_1, \dots, t_k , and so we write $t_1 \cdots t_k = s'_k(t_1, \dots, t_k)$.

Moreover, we have

$$\begin{aligned} s_1(t_1, \dots, t_k, 0, \dots, 0) &= t_1 + \dots + t_k = s'_1(t_1, \dots, t_k) \\ &\vdots \\ s_k(t_1, \dots, t_k, 0, \dots, 0) &= t_1 t_2 \cdots t_{k-1} t_k = s'_k(t_1, \dots, t_k) \end{aligned}$$

and $s_m(t_1, \dots, t_k, 0, \dots, 0) = 0$ for every m such that $m \geq k+1$.

Therefore, for the particular choice where we set $t_{k+1} = \dots = t_d = 0$ and leave t_1, \dots, t_k alone, we have

$$T(s_1, \dots, s_d) = T(s'_1, \dots, s'_k, 0, \dots, 0),$$

and hence that

$$c[s'_k]^2 = T(s'_1, \dots, s'_k, 0, \dots, 0).$$

Consequently

$$c[s'_k]^2 = \sum_{\alpha \in \Delta} b_\alpha (s'_1)^{\alpha_1} \cdots (s'_k)^{\alpha_k} (0)^{\alpha_{k+1}} \cdots (0)^{\alpha_{2k}}. \quad (3.4.7)$$

We will now want to utilise the *uniqueness* part of Theorem 3.4.1 in order to equate coefficients from both sides of equation (3.4.7) in order to see what monomial terms are actually present in the expression

$$\sum_{\alpha \in \Delta} c_\alpha s_1^{\alpha_1} \cdots s_{2k}^{\alpha_{2k}},$$

when $s_m = s_m(t_1, \dots, t_d)$ for each m such that $1 \leq m \leq 2k$. One can immediately see from the right-hand side of equation (3.4.7), that monomials corresponding

to tuples in Δ which have $\alpha_{j_0} \neq 0$ for some j_0 , with $k+1 \leq j_0 \leq 2k$, will automatically disappear under the particular choice of variables we have made. Hence the right-hand side of equation (3.4.7) reduces to

$$\sum_{\alpha \in \Delta'} c_{\alpha} (s'_1)^{\alpha_1} \cdots (s'_k)^{\alpha_k} \quad (3.4.8)$$

where $\Delta' = \{\alpha \in \Delta : \alpha_{k+1} + \cdots + \alpha_{2k} = 0\}$. We can now split Δ further by writing

$$\Delta = \Delta_1 \cup \Delta_2$$

where $\Delta_1 = \{\alpha \in \Delta : \sum_{j=1}^{k-1} \alpha_j = 0\}$ and $\Delta_2 = \{\alpha \in \Delta : \sum_{j=1}^{k-1} \alpha_j \geq 1\}$.

We observe next that there is only one tuple in Δ_1 , namely the one which has $\alpha_k = 2$ and all other entries equal to zero, and so, we see that (3.4.8) is actually equal to

$$c_{(0,\dots,0,2,0,\dots,0)} [s'_k]^2 + \sum_{\alpha \in \Delta_2} c_{\alpha} (s'_1)^{\alpha_1} \cdots (s'_k)^{\alpha_k}.$$

Therefore, going back to equation (3.4.7) we now can write

$$c[s'_k]^2 = c_{(0,\dots,0,2,0,\dots,0)} [s'_k]^2 + \sum_{\alpha \in \Delta_2} c_{\alpha} (s'_1)^{\alpha_1} \cdots (s'_k)^{\alpha_k}.$$

By comparing coefficients, the *uniqueness* part of Theorem 3.4.1 implies that we must have $c_{\alpha} = 0$ for all α in Δ_2 , and moreover that $c_{(0,\dots,0,2,0,\dots,0)} = c$.

Now, for a tuple $(\alpha_1, \dots, \alpha_d) \in \Delta$ having $\alpha_k \neq 2$, observe that it is not possible for α_k to satisfy $\alpha_k \geq 3$, for if it is, then the constraint condition $\sum_{j=1}^{2k} j\alpha_j = 2k$ is clearly violated, and so α_k must be either 0 or 1 in this case. Furthermore, if in addition to having $\alpha_k \neq 2$, the tuple also satisfies $\alpha_{k+1} + \cdots + \alpha_{2k} \geq 1$, then obviously α_k cannot be 1.

Therefore, since $\Delta = \Delta_1 \cup \{\alpha \in \Delta : \alpha_k \neq 2\}$ and $c_\alpha = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$ in Δ , we see from the above arguments that

$$Q(t_1, \dots, t_d) = cs_k^2 + \sum_{\Gamma} c_\alpha s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{2k}^{\alpha_{2k}} \quad (3.4.9)$$

where $\Gamma = \{\alpha \in \Delta : \alpha_{k+1} + \dots + \alpha_{2k} \geq 1\} \cap \{\alpha \in \Delta : \alpha_k = 0\}$.

The way to proceed next is clear, for we now have

$$|Q(t_1, \dots, t_d)| \geq c|s_k|^2 - \left| \sum_{\Gamma} c_\alpha s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{2k}^{\alpha_{2k}} \right|, \quad (3.4.10)$$

where of course now $s_n = s_n(t_1, \dots, t_d)$ for each n such that $1 \leq n \leq 2k$.

We set $t_1 = r_1, \dots, t_d = r_d$ and multiply through by $|\epsilon|^2$ in (3.4.10), and recalling that $|Q(\eta_1, \dots, \eta_d)| = |Q(r_1, \dots, r_d)|$, we can appeal to Lemma 3.4.2 in order to obtain

$$|\epsilon|^2 |\eta_1 - \eta_d|^2 \cdots |\eta_1 - \eta_{d-k+1}|^2 \gtrsim_d c |\epsilon s_k|^2 - \left| \sum_{\Gamma} c_\alpha \epsilon^2 s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{2k}^{\alpha_{2k}} \right|.$$

Note. Now we have $s_n = s_n(r_1, \dots, r_d)$ for each n such that $1 \leq n \leq 2k$.

If it happens to be the case that $c_\alpha = 0$ for every α in Γ then we are done, since we have by hypothesis that $|\epsilon s_k(r_1, \dots, r_d)| \sim 1$, and so, the desired conclusion follows trivially. So let us assume instead that we do have some of the coefficients c_α being nonzero for α belonging to Γ .

Our goal will now be to show that $\epsilon^2 s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{2k}^{\alpha_{2k}}$ can be made small for all $\alpha \in \Gamma$. Let us then consider a general monomial term $c_\alpha s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{2k}^{\alpha_{2k}}$ with $(\alpha_1, \dots, \alpha_{2k})$ in the set Γ . Observe that in a tuple α belonging to Γ there can be at most one

nonzero α_j for $j \geq k+1$ and if this is the case, it must have value equal to one. Hence, we can decompose the set Γ so that

$$\Gamma = \bigcup_{j=k+1}^{2k} \Gamma_j \quad (3.4.11)$$

where $\Gamma_j = \Delta \cap \{\alpha : \alpha_k = 0\} \cap \{\alpha : \alpha_j = 1\} \cap \{\alpha : \alpha_i = 0 \ \forall i \neq j : k+1 \leq i \leq 2k\}$.

If we consider each j in turn we see immediately that the result follows for the cases $k=1$ and $k=2$; since if $j=2k$ then on account of the constraint condition $\sum_{\ell=1}^{2k} \ell \alpha_\ell = 2k$ it follows that we must have $\alpha_1 = 0$, with $\alpha_2 = \dots = \alpha_k = 0$, and so for this tuple we have $s_1^{\alpha_1} s_2^{\alpha_2} \dots s_{2k}^{\alpha_{2k}} = s_{2k}$, and hence $\epsilon^2 s_{2k} = \epsilon(\epsilon s_{2k})$ is clearly small according to our hypotheses. Moreover, if $j=2k-1$ then in the same way we conclude that $s_1^{\alpha_1} s_2^{\alpha_2} \dots s_{2k}^{\alpha_{2k}} = s_1 s_{2k-1}$, and therefore, on account of our hypotheses, that $\epsilon^2 s_1 s_{2k-1} = (\epsilon s_1)(\epsilon s_{2k-1})$ is also small. However, if we consider $j=2k-2$, we now run into problems, as the power of ϵ is not big enough to match the weight of certain monomials which now begin to arise; and so we are not able to make such terms small.

However, if the reader recalls, we saw earlier that for certain types of tuple α , the corresponding monomial term cannot actually arise, as the coefficient c_α corresponding to the tuple α is zero. For instance, recall that $c_\alpha = 0$ for every $\alpha \in \Delta$ such that $\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$. So, we claim that the various troublesome terms do not in fact arise. To establish the claim, we will argue inductively.

All the arguments so far work for $k=1$ and $k=2$, and so, the first problematic case that we need to consider is $k=3$. Furthermore, the simplicity of this case, and the brevity of its analysis, makes it a perfect *apéritif* for illustrating the fundamental idea behind the approach in the general case. We revert again to

arbitrary variables t_1, \dots, t_d and consider (3.4.9) with Γ now decomposed as in (3.4.11). The strategy that we will employ is very similar to what we have done in the past, namely, we will set certain appropriate variables equal to zero and then by using the uniqueness of T , we will equate coefficients in order to see that certain monomials cannot arise as the coefficients associated to them are in fact zero.

The arguments so far imply that when $k = 3$ the polynomial T has the form

$$T(s_1, \dots, s_d) = cs_3^2 + a_1s_1^2s_4 + a_2s_2s_4 + a_3s_1s_5 + a_4s_6$$

where we have $c > 0$. Here one can clearly see that the only troublesome term is the monomial term $c_\alpha s_1^2 s_4$. The problem here is that it has weight equal to 3, and of course the power of ϵ that we have to multiply by is only 2. So we must see whether this term actually occurs, that is, we must prove that a_1 , which of course is $c_{(2,0,0,1,0,\dots,0)}$ in the general scheme, is zero.

We argue as follows. Setting $t_5 = t_6 = \dots = t_d = 0$ we obtain

$$Q(t_1, t_2, t_4, 0, \dots, 0) = T(s'_1, s'_2, s'_3, s'_4, 0, \dots, 0) = c(s'_3)^2 + a_1(s'_1)^2 s'_4 + a_2 s'_2 s'_4$$

where now s'_1, s'_2, s'_3 and s'_4 are the elementary symmetric polynomials of the variables t_1, t_2, t_3 and t_4 only. We then note that the monomial $t_1^3 t_2 t_3 t_4$ arises in $(s'_1)^2 s'_4$ but it clearly cannot arise in $(s'_3)^3$, $s'_2 s'_4$, or $Q(t_1, t_2, t_3, t_4, 0, \dots, 0)$. Hence it follows that a_1 must be zero.

Consequently, we are then left with

$$Q(t_1, t_2, t_4, 0, \dots, 0) = T(s'_1, s'_2, s'_3, s'_4, 0, \dots, 0) = c(s'_3)^2 + a_2 s'_2 s'_4,$$

and reverting to the arbitrary variables t_1, \dots, t_d , we therefore see that T has the form

$$T(s_1, \dots, s_d) = cs_3^2 + b_1s_4s_2 + b_2s_5s_1 + b_3s_6.$$

Hence, if we follow the above example, we can in general, for $k \geq 3$, utilise the way that Γ is decomposed in (3.4.11) to write

$$T(s_1, \dots, s_d) = cs_k^2 + \sum_{j=1}^{k-2} s_{k+j} T_j(s_1, \dots, s_{k-j}) + b_{k-1}s_1s_{2k-1} + b_k s_{2k} \quad (3.4.12)$$

where $c > 0$ and for each j such that $1 \leq j \leq k-2$

$$T_j(s_1, \dots, s_{k-j}) = \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_{k-j}) \\ \alpha_1+2\alpha_2+\dots+(k-j)\alpha_{k-j}=k-j}} b_{\alpha}^j s_1^{\alpha_1} \dots s_{k-j}^{\alpha_{k-j}}. \quad (3.4.13)$$

In (3.4.12) we have of course chosen to denote $c_{(1,0,\dots,0,1,0)}$ as b_{k-1} and $c_{(0,\dots,0,1)}$ as b_k . Now, for each j such that $1 \leq j \leq k-2$, let us denote by Δ_{k-j} the set of tuples over which the sum in (3.4.13) is taken.

Our goal now is to show for each j such that $1 \leq j \leq k-2$ that $b_{\alpha}^j = 0$ for every $\alpha \in \Delta_{k-j}$ except for that α with all entries equal to zero save $\alpha_{k-j} = 1$; in this case, when we multiply by $|\epsilon|^2$ later on, we will have $\epsilon^2 s_{k+j} s_{k-j}$, which of course can be made small. We will do this inductively and so we will need to impose an ordering on the $(k-j)$ -tuples arising in each sum defining T_j . We do this as follows. Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ be two general distinct tuples. We say that $\alpha < \beta$ if there is an ℓ , such that $1 \leq \ell \leq n$, so that

$$\alpha_r + \dots + \alpha_n = \beta_r + \dots + \beta_n$$

for every r such that $1 \leq r \leq \ell-1$ but $\alpha_{\ell} + \dots + \alpha_n < \beta_{\ell} + \dots + \beta_n$. Given any

two distinct n -tuples α and β , it is a trivial matter to see that either $\alpha < \beta$ or $\alpha > \beta$.

Given the above ordering, we can now impose it on the set Δ_{k-j} in order to decompose it as follows. Let β_0^j denote the tuple in Δ_{k-j} having all entries equal to zero except for the $(k-j)$ -th entry, which is equal to 1. We write

$$\Delta_{k-j} = \Theta_j \cup \{\beta_0^j\} \cup \tilde{\Theta}_j$$

where $\Theta_j = \{\alpha \in \Delta_{k-j} : \alpha < \beta_0^j\}$ and $\tilde{\Theta}_j = \{\alpha \in \Delta_{k-j} : \alpha > \beta_0^j\}$.

We are now in a position to proceed with proving our goal. Let us consider the case $j = 1$ first. Our aim here is to show that $T_1(s_1, \dots, s_{k-1}) = b_1 s_{k+1} s_{k-1}$ where b_1 is a constant. Starting with the tuples α in the set Θ_1 , we proceed by considering each b_α^1 one by one and show that it is equal to zero. We start with the maximal tuple, which we denote as $\hat{\alpha}$. Observe that $\hat{\alpha} = (k-1, 0, \dots, 0)$. This follows because all other $\alpha = (\alpha_1, \dots, \alpha_{k-1}) \neq \hat{\alpha}$ satisfy

$$\alpha_2 + 2\alpha_3 + \dots + (k-1)\alpha_{k-1} \geq 1$$

and so we have

$$\begin{aligned} \alpha_1 + \dots + \alpha_{k-1} &= \alpha_1 + 2\alpha_2 + \dots + (k-1)\alpha_{k-1} - (\alpha_2 + 2\alpha_3 + \dots + (k-2)\alpha_{k-1}) \\ &= (k-1) - (\alpha_2 + 2\alpha_3 + \dots + (k-2)\alpha_{k-1}) \\ &< k-1 \\ &= \hat{\alpha}_1 + \dots + \hat{\alpha}_{k-1} \end{aligned}$$

for every $\alpha \neq \hat{\alpha}$, where $\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1}) = (k-1, 0, \dots, 0)$.

To show that $b_{(k-1,0,\dots,0)}^1 = 0$ we set $t_{k+2} = \dots = t_d = 0$, and thus

$$Q(t_1, \dots, t_{k+1}, 0, \dots, 0) = cs_k^2 + s_{k+1}T_1(s_1, \dots, s_{k-1}).$$

Note. For the sake of notational convenience, we are abusing the notation above by writing s_j in instead of the usual s'_j . The reader should be aware that the s_j above are actually the elementary symmetric polynomials of the variables t_1, \dots, t_{k+1} . From now on we will continue to do this abuse of notation whenever we set certain variables equal to zero, and we therefore ask the reader to be aware of this convention whenever such future situations will arise.

We simply observe that the monomial $t_1^k t_2 \dots t_{k+1}$ which arises in $s_1^{k-1} s_{k+1}$ does not arise in s_k^2 , or any other term $s_1^{\alpha_1} \dots s_{k-1}^{\alpha_{k-1}} s_{k+1}$ appearing in $s_{k+1}T_1(s_1, \dots, s_{k-1})$, since for all $\alpha = (\alpha_1, \dots, \alpha_{k-1})$ belonging to Θ_1 , such that $\alpha < \hat{\alpha}$, we have $k > \alpha_1 + \dots + \alpha_{k-1} + 1$. Moreover, since $k \geq 3$, the monomial $t_1^k t_2 \dots t_{k+1}$ cannot arise in $Q(t_1, \dots, t_{k+1}, 0, \dots, 0)$. Hence $b_{(k-1,0,\dots,0)}^1 = 0$.

Next, we consider the tuple which is immediately below $\hat{\alpha}$ in terms of the ordering that we have imposed, call it $\beta = (\beta_1, \dots, \beta_{k-1})$, say. We observe that the monomial

$$t_1^{\beta_1 + \dots + \beta_{k-1} + 1} t_2^{\beta_2 + \dots + \beta_{k-1} + 1} \dots t_{k-1}^{\beta_{k-1} + 1} t_k t_{k+1} \quad (3.4.14)$$

which arises in $s_1^{\beta_1} \dots s_{k-1}^{\beta_{k-1}} s_{k+1}$ cannot arise in s_k^2 , or any other term $s_1^{\alpha_1} \dots s_{k-1}^{\alpha_{k-1}} s_{k+1}$ appearing in $s_{k+1}T_1(s_1, \dots, s_{k-1})$, where $\beta > \alpha$. This follows precisely because $\beta > \alpha$ implies that there is an ℓ satisfying $1 \leq \ell \leq k-1$ so that

$$\beta_\ell + \dots + \beta_{k-1} + 1 > \alpha_\ell + \dots + \alpha_{k-1} + 1,$$

and so the monomial in (3.4.14), since it contains the term $t_\ell^{\beta_\ell + \dots + \beta_{k-1} + 1}$, therefore cannot arise in any other term $s_1^{\alpha_1} \dots s_{k-1}^{\alpha_{k-1}} s_{k+1}$ appearing in $s_{k+1}T_1(s_1, \dots, s_{k-1})$.

Moreover, the monomial in (3.4.14) cannot also arise in $Q(t_1, \dots, t_{k+1}, 0, \dots, 0)$ since $\beta > \beta_0^1$ implies there exists an ℓ' satisfying $1 \leq \ell' \leq k-1$ so that

$$\beta_{\ell'} + \dots + \beta_{k-1} + 1 \geq 3,$$

and so we conclude that $b_\beta^1 = 0$. We continue repeating this process in an inductive manner for all of the remaining tuples in Θ_1 , and so we conclude that $b_\alpha^1 = 0$ for every α in Θ_1 .

We now apply the same approach to the set of tuples in $\tilde{\Theta}_1$. Starting with the first tuple β in $\tilde{\Theta}_1$ such that $\beta < \beta_0^1$, we can see in exactly the same way as we did above that the monomial

$$t_1^{\beta_1 + \dots + \beta_{k-1} + 1} t_2^{\beta_2 + \dots + \beta_{k-1} + 1} \dots t_{k-1}^{\beta_{k-1} + 1} t_k t_{k+1} \quad (3.4.15)$$

which arises in $s_1^{\beta_1} \dots s_{k-1}^{\beta_{k-1}} s_{k+1}$ cannot arise in s_k^2 , or any other term $s_1^{\alpha_1} \dots s_{k-1}^{\alpha_{k-1}} s_{k+1}$ appearing in $s_{k+1} T_1(s_1, \dots, s_{k-1})$, where $\beta > \alpha$.

Moreover, the monomial in (3.4.15), cannot also arise in $Q(t_1, \dots, t_{k+1}, 0, \dots, 0)$ since $\beta < \beta_0^1$ implies that there exists an ℓ' satisfying $1 \leq \ell' \leq k-1$ so that

$$\beta_{\ell'} + \dots + \beta_{k-1} < 1,$$

which in turn implies that we must have $\beta_{\ell'} = \dots = \beta_{k-1} = 0$, and hence that $\beta_n + \dots + \beta_{k-1} + 1 = 1$ for each n such that $\ell' \leq n \leq k-1$. Thus the monomial in (3.4.15), since it contains the term $t_{\ell'}^1 \dots t_{k-1}^1$, cannot therefore arise in $Q(t_1, \dots, t_{k+1}, 0, \dots, 0)$. Hence we conclude that $b_\beta^1 = 0$. We continue repeating this process in an inductive manner for all of the remaining tuples in $\tilde{\Theta}_1$, and so

we conclude that $b_\alpha^1 = 0$ for every α in $\tilde{\Theta}_1$. Hence, we finally arrive at the desired conclusion that $T_1(s_1, \dots, s_{k-1}) = b_1 s_{k-1}$, where we have of course set $b_{\beta_0^1}^1$ equal to b_1 .

We then move on to the polynomial T_2 and set $t_{k+3} = \dots = t_d = 0$, thus we have

$$Q(t_1, \dots, t_{k+2}, 0, \dots, 0) = cs_k^2 + b_1 s_{k-1} s_{k+1} + s_{k+2} T_2(s_1, \dots, s_{k-2})$$

since now $s_j = 0$ for every $j \geq k+3$.

Note. We are again abusing notation here, the symmetric polynomials are now in fact polynomial functions of the variables t_1, \dots, t_{k+2} .

In a similar way to the case $j = 1$ one shows that $T_2(s_1, \dots, s_{k-2}) = b_2 s_{k-2}$, but instead of carrying on with the details of the case $j = 2$ we will demonstrate the general inductive procedure for $j = m$ where m is any integer such that $2 \leq m \leq k-2$. The overall pattern is very similar to the case where $j = 1$, and so we will keep the details brief in order to avoid unnecessary repetition.

The procedure is as follows. Suppose that we have shown case by case that $T_j(s_1, \dots, s_{k-j}) = b_j s_{k-j}$ for every j such that $1 \leq j \leq m-1$ and that we now wish to prove that $T_m(s_1, \dots, s_{k-m}) = b_m s_{k-m}$. We consider the polynomial $s_{k+m} T_m(s_1, \dots, s_{k-m})$, which we know takes the form

$$s_{k+m} \sum_{\alpha \in \Delta_{k-m}} b_\alpha^m s_1^{\alpha_1} \dots s_{k-m}^{\alpha_{k-m}}.$$

Our goal is of course to show that $b_\alpha^m = 0$ for every tuple α in Δ_{k-m} except for the tuple β_0^m , which gives rise to the non-offending monomial term $s_{k+m} s_{k-m}$. In the same way as before we consider the decomposed form of Δ_{k-m} where we have

$$\Delta_{k-m} = \Theta_m \cup \{\beta_0^m\} \cup \tilde{\Theta}_m$$

with $\Theta_m = \{\alpha \in \Delta_{k-m} : \alpha < \beta_0^m\}$ and $\tilde{\Theta}_m = \{\alpha \in \Delta_{k-m} : \alpha > \beta_0^m\}$.

As before we start with the maximal tuple $\hat{\alpha}$ in Θ_m . One observes in the same way as in the case $j = 1$ that the maximal tuple $\hat{\alpha}$ in Θ_m is $(k - m, 0, \dots, 0)$. To observe that $b_{\hat{\alpha}}^m = 0$ we set $t_{k+m+1} = \dots = t_d = 0$ and thus, along with the induction assumption that $T_j(s_1, \dots, s_{k-j}) = b_j s_{k-j}$ for every j such that $1 \leq j \leq m - 1$, we obtain

$$Q(t_1, \dots, t_{k+m}, 0, \dots, 0) = cs_k^2 + \sum_{j=1}^{m-1} b_j s_{k-j} s_{k+j} + s_{k+m} T_m(s_1, \dots, s_{k-m}).$$

Note. The above symmetric polynomials are now symmetric polynomials of the variables t_1, \dots, t_{k+m} .

By noting the maximality of the tuple $(k-m, 0, \dots, 0)$, we observe in the same way as before that the monomial $t_1^{k-m+1} t_2 \cdots t_{k+m}$ which arises in $s_1^{k-m} s_{k+m}$ cannot arise in $(s_k)^2, (s_{k-1} s_{k+1}), \dots, (s_{k-m+1} s_{k+m+1})$, or any other term $s_1^{\alpha_1} \cdots s_{k-m}^{\alpha_{k-m}} s_{k+m}$ appearing in $s_{k+m} T_m(s_1, \dots, s_{k-m})$. Moreover, since $m \leq k - 2$ we have that $k - m + 1 \geq 3$, and so we see that the monomial $t_1^{k-m+1} t_2 \cdots t_{k+m}$ cannot arise in $Q(t_1, \dots, t_{k+m}, 0, \dots, 0)$. Hence it follows that we must have $b_{\hat{\alpha}}^m = 0$.

Next, we consider the tuple which is immediately below $\hat{\alpha}$ and we observe in the same way as before that the monomial

$$t_1^{\beta_1 + \dots + \beta_{k-m} + 1} t_2^{\beta_2 + \dots + \beta_{k-m} + 1} \cdots t_{k-m}^{\beta_{k-m} + 1} t_{k-m+1} \cdots t_k t_{k+1} \cdots t_{k+m} \quad (3.4.16)$$

which arises in $s_1^{\beta_1} \cdots s_{k-m}^{\beta_{k-m}} s_{k+m}$ cannot arise in $(s_k)^2, (s_{k-1} s_{k+1}), \dots, (s_{k-m+1} s_{k+m+1})$, or any other term $s_1^{\alpha_1} \cdots s_{k-m}^{\alpha_{k-m}} s_{k+m}$ appearing in $s_{k+m} T_m(s_1, \dots, s_{k-m})$, where $\beta > \alpha$.

Moreover, the monomial in (3.4.16) cannot also arise in $Q(t_1, \dots, t_{k+m}, 0, \dots, 0)$

since $\beta > \beta_0^m$ implies that there exists an ℓ' satisfying $1 \leq \ell' \leq k - m$ so that

$$\beta_{\ell'} + \dots + \beta_{k-m} + 1 \geq 3,$$

and hence we conclude that $b_\beta^m = 0$. We continue repeating this process in an inductive manner for all the remaining tuples in Θ_m and thus conclude that $b_\alpha^m = 0$ for every α in Θ_m .

The same approach as above is applied to the set of tuples in $\tilde{\Theta}_m$. Starting with the first tuple β in $\tilde{\Theta}_1$ such that $\beta < \beta_0^m$, we can see in exactly the same way as we did above that the monomial

$$t_1^{\beta_1 + \dots + \beta_{k-m} + 1} t_2^{\beta_2 + \dots + \beta_{k-m} + 1} \dots t_{k-m}^{\beta_{k-m} + 1} t_{k-m+1} \dots t_k t_{k+1} \dots t_{k+m} \quad (3.4.17)$$

which arises in $s_1^{\beta_1} \dots s_{k-m}^{\beta_{k-m}} s_{k+m}$ cannot arise in $(s_k)^2, (s_{k-1} s_{k+1}), \dots, (s_{k-m+1} s_{k+m+1})$, or any other term $s_1^{\alpha_1} \dots s_{k-m}^{\alpha_{k-m}} s_{k+m}$ appearing in $s_{k+m} T_m(s_1, \dots, s_{k-m})$, where $\beta > \alpha$.

Moreover, the monomial in (3.4.17), cannot also arise in $Q(t_1, \dots, t_{k+m}, 0, \dots, 0)$ since $\beta < \beta_0^m$ implies that there exists an ℓ' satisfying $1 \leq \ell' \leq k - m$ so that

$$\beta_{\ell'} + \dots + \beta_{k-m} < 1,$$

which in turn implies that we must have $\beta_{\ell'} = \dots = \beta_{k-m} = 0$, and hence that $\beta_n + \dots + \beta_{k-m} + 1 = 1$ for each n such that $\ell' \leq n \leq k - m$. Thus the monomial in (3.4.17), since it contains the term $t_{\ell'}^1 \dots t_{k-m}^1$, cannot therefore arise in $Q(t_1, \dots, t_{k+m}, 0, \dots, 0)$. Hence we conclude that $b_\beta^m = 0$. We continue repeating this process in an inductive manner for all of the remaining tuples in $\tilde{\Theta}_m$, and so we conclude that $b_\alpha^m = 0$ for every α in $\tilde{\Theta}_m$. Hence, we finally arrive at the

desired conclusion that $T_m(s_1, \dots, s_{k-m}) = b_m s_{k-m}$, where we have of course in the same way as before set $b_{\beta_0^m}^m$ equal to b_m .

Therefore, via this inductive procedure we finally conclude that

$$T(s_1, \dots, s_d) = c s_k^2 + b_1 s_{k-1} s_{k+1} + b_2 s_{k-2} s_{k+2} + \dots + b_{k-1} s_1 s_{2k-1} + b_k s_{2k}$$

where $c > 0$ and the symmetric polynomials s_j above are functions of the variables t_1, \dots, t_d .

Hence, since $Q(t_1, \dots, t_d) = T(s_1(t_1, \dots, t_d), \dots, s_d(t_1, \dots, t_d))$, we now have

$$|Q(t_1, \dots, t_d)| \geq c |s_k|^2 - \sum_{j=1}^k |b_j| |s_{k-j}| |s_{k+j}|.$$

Therefore, setting $t_1 = r_1, \dots, t_d = r_d$ and multiplying through by $|\epsilon|^2$ in the above, and applying the hypotheses that $|\epsilon s_i(r_1, \dots, r_d)| \leq 1$ for each i such that $k+1 \leq i \leq d$ with $|\epsilon s_k(r_1, \dots, r_d)|$ being about 1, we see that

$$\begin{aligned} |\epsilon^2 Q(r_1, \dots, r_d)| &\geq c |\epsilon s_k|^2 - \sum_{j=1}^k |b_j| |\epsilon s_{k-j}| |\epsilon s_{k+j}| \\ &\gtrsim c - \sum_{j=1}^k |b_j| |\epsilon s_{k-j}|. \end{aligned}$$

Hence, under the hypotheses that the constants

$$|\epsilon|, |\epsilon s_1(r_1, \dots, r_d)|, \dots, |\epsilon s_{k-1}(r_1, \dots, r_d)|$$

are small, it thus follows that $|\epsilon^2 Q(r_1, \dots, r_d)|$ is bounded below by $O(1)$.

Now, if the reader recalls, we have already observed that

$$|\eta_1 - \eta_d|^2 \dots |\eta_1 - \eta_{d-k+1}|^2 \gtrsim_d |Q(\eta_1, \dots, \eta_d)| = |Q(r_1, \dots, r_d)|.$$

Therefore, we can finally conclude that

$$|\epsilon| |\eta_1 - \eta_d| \dots |\eta_1 - \eta_{d-k+1}| \gtrsim_d 1,$$

and so, the proof is complete. □

3.5 Sharpness of the restriction $d \geq 2k$

We saw that the combinatorial inequality

$$1 \lesssim_d |\epsilon| |t_1 - t_{d-k+1}| \dots |t_1 - t_d|$$

plays a key role when one tries to establish the basic uniform set inclusion of Theorem 3.3.7. Moreover, prior to the proof of Theorem 3.3.7, we remarked that the restriction $d \geq 2k$ is sharp, and that the sharpness is to be taken in the sense that if a polynomial P has degree $d < 2k$ then the basic uniform set inclusion of Theorem 3.3.7 will fail to be true. We mentioned that this can be seen by considering the polynomial $P(t) = \epsilon t^{k-1}(t - r)^k$, where $r = \epsilon^{-1/k}$ with ϵ small, and observing that there are no uniform estimates for $|S_{\delta, P}|$ in this case.

In fact, an alternative way to view the sharpness of the restriction $d \geq 2k$, is to consider the above combinatorial inequality as follows. Consider two clusters A and B of points on the real line with cardinalities $|A|$ and $|B|$, and think of the

points within each cluster to be very close to one another but the two clusters to be very far apart. We will in fact see in a moment that we can actually take each cluster to just consist of a single point, as long as each point in the respective sets A and B will have the right multiplicity.

Now, for any point in cluster A , we start to compute k distances from it, and to keep these distances large we would of course always want to use points from B . However, if k is greater than $|B|$ this would force us to use a distance within A which is very small. Therefore if $k > \min(|A|, |B|)$ we will have a chance to get a contradiction to the key combinatorial inequality above. Note that $d = |A| + |B|$ and therefore $k > d/2$ is likely to give us a counterexample, and indeed this is the case if one simply considers

$$P(t) = \epsilon t^{k-1}(t - r)^k$$

where $r = \epsilon^{-1/k}$ and $\deg(P) = 2k - 1$. Here we see that the elementary symmetric polynomials of $t_1 = \dots = t_k = r, t_{k+1} = \dots = t_d = 0$ satisfy

$$|\epsilon s_j| \lesssim |\epsilon|^{\frac{k-j}{k}}$$

for each j such that $0 \leq j \leq k$, where \lesssim improves to $=$ when $j = k$, and $|\epsilon s_j| = 0$ for every j such that $j \geq k+1$. Now, from the above discussion, we clearly require the two roots r and 0 to be very far apart, and so we must take $|\epsilon| \ll 1$. Hence, we will have that $|\epsilon|^{\frac{k-j}{k}}$ is small for each j such that $0 \leq j \leq k-1$, and moreover, along with having $|\epsilon s_j| = 0$ for every j such that $j \geq k+1$, we see that this case satisfies the condition we used in the proof of the set inclusion theorem, namely that $|\epsilon|, |\epsilon s_1|, \dots, |\epsilon s_{k-1}|$ are small, $|\epsilon s_k| = 1$, and $|\epsilon s_j| \leq 1$ for every j such that $j \geq k+1$.

3.6 Application of the sub-level set inclusion theorem to the setting of multilinear sub-level set operators

We now turn to formulating the analogue of Theorem 3.3.7 in the context of multilinear sub-level set operators. The theorem regarding the *structure* of global sub-level sets $S_{\delta,P}$ can be “bootstrapped” to a theorem about uniform estimates for the multilinear sub-level set operator $S_{\delta}^{P(\Phi),K,\pi}$, given an a priori estimate for the multilinear oscillatory integral operators $\Lambda_{\lambda}^{\Phi,K,\pi}$.

Theorem 3.6.1. *For any $k \geq 0$ let P be a normalised real polynomial $P(t) = \sum_{m=0}^d c_m t^m$ with $d \geq 2k$ and $|c_{d-k}| = 1$. If we assume a priori the estimate*

$$|\Lambda_{\lambda}^{\Phi,K,\pi}(f_1, \dots, f_L)| \leq A|\lambda|^{-\alpha} \prod_{i=1}^L \|f_i\|_{L^{p_i}(\mathbf{R}^{m_i})}, \quad (3.6.1)$$

then the following estimate

$$|S_{\delta}^{P(\Phi),K,\pi}(f_1, \dots, f_L)| \leq AC_{\alpha,d} \left\{ \begin{array}{ll} \delta^{\alpha/d} & \alpha < 1, \\ \delta^{1/d} \log\left(\frac{1}{\delta}\right) & \alpha = 1, \\ \delta^{1/d} & \alpha > 1, \end{array} \right\} \prod_{i=1}^L \|f_i\|_{L^{p_i}(\mathbf{R}^{m_i})}$$

holds whenever $0 \leq \delta < 1$.

Remark. If we start a priori with a bound on the operator $S_{\delta}^{\Phi,K,\pi}$, then there is no immediate formal implication that one can use in order to obtain a bound on the operator $S_{\delta}^{P(\Phi),K,\pi}$. Nevertheless, in the proof of the above result that we shall give next, one can see that we could also just as well assume a priori the uniform data

$$|S_{\eta}^{\Phi-c,K,\pi}(f_1, \dots, f_L)| \leq B\eta^a \prod_{i=1}^L \|f_i\|_{L^{p_i}(\mathbf{R}^{m_i})}$$

and go on to bound the sub-level set operator $S_\delta^{P(\Phi), K, \pi}$. However, it was precisely because of the fact that oscillatory integral estimates remain invariant under translations of the phase by any constant, that we considered studying sets such as $\{x \in K : |\Phi(x) - c| < \eta\}$, and consequently, operators such as $S_\eta^{\Phi - c, K, \pi}$, in the first place.

Proof. Since oscillatory integral estimates are invariant under centred monomial transformations of the phase, our task therefore will be to apply Theorem 3.3.7 in order to bound the operator

$$S_\delta^{P(\Phi), K, \pi}(f_1, \dots, f_L) = \int_{\{x \in K : |P(\Phi(x))| < \delta\}} \prod_{i=1}^L f_i(\pi_i(x)) dx$$

by a finite sum of simpler sub-level set operators $S_{\delta_d}^{\Phi - r, K, \pi}(f_1, \dots, f_L)$ so that we can then apply the folklore procedure to each one individually. The constant r will turn out to be, in general, the real part of a particular root of P , and $\delta_d = A_d \delta^{1/d}$. The invariance of oscillatory integral estimates under centred monomial transformations of the phase will make each corresponding estimate of $S_{\delta_d}^{\Phi - r, K, \pi}(f_1, \dots, f_L)$ uniform in r , and so summing everything up will complete the proof.

Keeping these things in mind we proceed as follows. We have the set identity

$$\{x \in K : |P(\Phi(x))| < \delta\} = K \cap \Phi^{-1}(\mathbf{R} \cap \{t : |P(t)| < \delta\}).$$

For every $\delta > 0$ such that $\delta < 1$, Theorem 3.3.7 implies that

$$\mathbf{R} \cap \{t : |P(t)| < \delta\} \subset \bigcup_{\xi \in \mathcal{R}_P} \{t \in \mathbf{R} : |t - \Re(\xi)| \leq A_d \delta^{1/d}\},$$

and this in turn then implies that

$$\begin{aligned}
\Phi^{-1}(\mathbf{R} \cap \{t : |P(t)| < \delta\}) &\subset \Phi^{-1} \left(\bigcup_{\xi \in \mathcal{R}_P} \{t \in \mathbf{R} : |t - \Re(\xi)| \leq A_d \delta^{1/d}\} \right) \\
&= \bigcup_{\xi \in \mathcal{R}_P} \{x : |\Phi(x) - \Re(\xi)| \leq A_d \delta^{1/d}\}.
\end{aligned}$$

Consequently, putting all the above things together, we therefore obtain the uniform set inclusion

$$\{x \in K : |P(\Phi(x))| < \delta\} \subset \bigcup_{\xi \in \mathcal{R}_P} \{x \in K : |\Phi(x) - \Re(\xi)| \leq A_d \delta^{1/d}\}.$$

Hence, we see that

$$\begin{aligned}
\int_{\{x \in K : |P(\Phi(x))| < \delta\}} \prod_{i=1}^L f_i(\pi_i(x)) dx &= \int_{K \cap \Phi^{-1}(\mathbf{R} \cap \{t : |P(t)| < \delta\})} \prod_{i=1}^L f_i(\pi_i(x)) dx \\
&\leq \sum_{\xi \in \mathcal{R}_P} \int_{\{x \in K : |\Phi(x) - \Re(\xi)| \leq A_d \delta^{1/d}\}} \prod_{i=1}^L f_i(\pi_i(x)) dx \\
&= \sum_{\xi \in \mathcal{R}_P} \int_K \chi_{\{x \in K : |\Phi(x) - \Re(\xi)| \leq A_d \delta^{1/d}\}}(x) \prod_{i=1}^L f_i(\pi_i(x)) dx.
\end{aligned}$$

Let us denote the set $\{x \in K : |\Phi(x) - \Re(\xi)| \leq A_d \delta^{1/d}\}$ by K_λ , and construct Ψ even with $\Psi \in \mathcal{S}(\mathbf{R})$, so that point-wise we have the inequality

$$\chi_{K_\lambda}(x) \leq \Psi \left(\frac{\Phi(x) - \Re(\xi)}{A_d \delta^{1/d}} \right).$$

We now have

$$\begin{aligned}
\int_K \chi_{K_\lambda}(x) \prod_{i=1}^L f_i(\pi_i(x)) dx &\leq \int_K \Psi \left(\frac{\Phi(x) - \Re(\xi)}{A_d \delta^{1/d}} \right) \prod_{i=1}^L f_i(\pi_i(x)) dx \\
&= \int_{-\infty}^{\infty} \int_K e^{i c'_d s \delta^{-1/d} [\Phi(x) - \Re(\xi)]} \prod_{i=1}^L f_i(\pi_i(x)) dx \widehat{\Psi}(s) ds \\
&= 2\Re \left(\int_0^{\infty} \int_K e^{i c'_d s \delta^{-1/d} [\Phi(x) - \Re(\xi)]} \prod_{i=1}^L f_i(\pi_i(x)) dx \widehat{\Psi}(s) ds \right)
\end{aligned}$$

where $c'_d = \frac{2\pi}{A_d}$.

Moreover

$$\begin{aligned}
\left| \int_K e^{i c'_d s \delta^{-1/d} [\Phi(x) - \Re(\xi)]} \prod_{i=1}^L f_i(\pi_i(x)) dx \right| &= \left| \int_K e^{i c'_d s \delta^{-1/d} \Phi(x)} \prod_{i=1}^L f_i(\pi_i(x)) dx \right| \\
&= |\Lambda_{c'_d s \delta^{-1/d}}^{\Phi, K, \pi}(f_1, \dots, f_L)|.
\end{aligned}$$

Hence, applying also (3.6.1), we now have

$$\begin{aligned}
|S_\delta^{P(\Phi), K, \pi}(f_1, \dots, f_L)| &\leq 2 \sum_{\xi \in \mathcal{R}_P} \int_0^{\infty} |\Lambda_{c'_d s \delta^{-1/d}}(f_1 \dots f_L)| |\widehat{\Psi}(s)| ds \\
&\leq 2 \sum_{\xi \in \mathcal{R}_P} A \int_0^{\infty} \min(1, c_d'^{-\alpha} \delta^{\alpha/d} s^{-\alpha}) |\widehat{\Psi}(s)| ds \prod_{i=1}^L \|f_i\|_{L^{p_i}(\mathbf{R}^{m_i})}.
\end{aligned}$$

One can then compute the estimate

$$\int_0^{\infty} \min(1, c_d'^{-\alpha} \delta^{\alpha/d} s^{-\alpha}) |\widehat{\Psi}(s)| ds \leq C_{\alpha, d} \begin{cases} \delta^{\alpha/d} & \alpha < 1, \\ \delta^{1/d} \log\left(\frac{1}{\delta}\right) & \alpha = 1, \\ \delta^{1/d} & \alpha > 1, \end{cases}$$

to finally conclude that

$$|S_\delta^{P(\Phi), K, \pi}(f_1, \dots, f_L)| \leq AC_{\alpha, d} \left\{ \begin{array}{ll} \delta^{\alpha/d} & \alpha < 1, \\ \delta^{1/d} \log\left(\frac{1}{\delta}\right) & \alpha = 1, \\ \delta^{1/d} & \alpha > 1, \end{array} \right\} \prod_{i=1}^L \|f_i\|_{L^{p_i}(\mathbf{R}^{m_i})}.$$

Note. We have subsumed the constant $\sum_{\xi \in \mathcal{R}_P} 1$, which is the number of roots according to multiplicity of the polynomial P , into the final constant $C_{\alpha, d}$.

□

Having established the mapping properties of the generalised sub-level set operator in Theorem 3.6.1, we can now deduce the measure estimate of the corresponding sub-level set as a simple consequence of it.

Corollary 3.6.2. *For any $k \geq 0$ let P be a normalised real polynomial $P(t) = \sum_{m=0}^d c_m t^m$ with $d \geq 2k$ and $|c_{d-k}| = 1$. If we assume a priori the estimate*

$$|\Lambda_\lambda^{\Phi, K, \pi}(f_1, \dots, f_L)| \leq A|\lambda|^{-\alpha} \prod_{i=1}^L \|f_i\|_{L^{p_i}(\mathbf{R}^{m_i})},$$

then the following estimate

$$|\{x \in K : |P(\Phi(x))| < \delta\}| \leq AC_{\alpha, d} \left\{ \begin{array}{ll} \delta^{\alpha/d} & \alpha < 1, \\ \delta^{1/d} \log\left(\frac{1}{\delta}\right) & \alpha = 1, \\ \delta^{1/d} & \alpha > 1, \end{array} \right.$$

holds whenever $0 \leq \delta < 1$.

Proof. Take $\pi_j = i_d$ and $f_j = \chi_K$ for each $j \in \{1, \dots, L\}$, then

$$|S_\delta^{P(\Phi), K, \pi}(\chi_K, \dots, \chi_K)| = |\{x \in K : |P(\Phi(x))| < \delta\}|.$$

The estimate follows by applying Theorem 3.6.1.

□

Remark. We incur a factor of $\prod_{j=1}^L \|\chi_K\|_{L^{p_j}(\mathbf{R}^{n_j})}$ in the constant $C_{\alpha,d}$.

We can now apply Corollary 3.6.2 to study the stability of sub-level set estimates.

3.7 Stability of sub-level set estimates

We saw in Section 3.2 that under small polynomial perturbations of the phase Φ that the oscillatory integral estimate as given in the paper of D. H. Phong and E. M. Stein [34] was not uniform. However, we alluded to the fact that the situation for sub-level set operator estimates fairs much better. Our next result shows this to indeed be the case. For $\delta > 0$, let us denote by $E_{\delta,\Phi}$ the sub-level set associated to the real analytic phase Φ defined on $\mathbf{R} \times \mathbf{R}$ so that

$$E_{\delta,\Phi} = \{(x, y) \in K : |\Phi(x, y)| < \delta\}$$

where K is an arbitrary compact set contained in $\mathbf{R} \times \mathbf{R}$. The associated sub-level set operator is of course denoted by S_{δ}^{Φ} . We can put Theorem 3.6.1 along with Theorem 3.2.1 from [34] to work to obtain, as an immediate consequence, the following stability result for sub-level set operators.

Proposition 3.7.1. *Let $\Phi(x, y)$ be a real analytic phase function defined on $\mathbf{R} \times \mathbf{R}$, and let $P_{\epsilon}(t) = t^2 - \epsilon t$ for $0 < \epsilon < 1$. Then the following sharp L^2 estimate holds*

$$\|S_{\delta}^{P_{\epsilon} \circ \Phi}\|_{L^2 \rightarrow L^2} \leq AC_{\gamma} \begin{cases} \delta^{\gamma/4} & 0 \leq \gamma < 2, \\ \delta^{1/2} \log\left(\frac{1}{\delta}\right) & \gamma = 2, \\ \delta & \gamma > 2, \end{cases}$$

where A , and C_{γ} are absolute constants and γ is the Newton decay rate of Φ . Moreover, we also have the exact same estimate for the sub-level set operator $S_{\delta}^{\Phi^2}$ as well.

Proof. We simply take the L^2 operator norm estimate for the oscillatory integral operator T_λ in Theorem 3.2.1 as our a priori oscillatory integral estimate and apply Theorem 3.6.1. \square

So we see that as $\epsilon \rightarrow 0$ the sub-level set operator estimate for $S_\delta^{P_\epsilon \circ \Phi}$ agrees with that of $S_\delta^{\Phi^2}$. Hence, we observe, that sub-level set operator estimates are uniform and behave well under small perturbations of the phase Φ . However, as we have seen in Section 3.2, oscillatory integral L^2 norm estimates where the exponent arises from the Newton decay rate are unfortunately not uniform for small perturbations of the phase.

Chapter 4

A calculus of one dimensional oscillatory estimates in the setting of asymptotic expansions

4.1 Introduction

Throughout the whole thesis so far we have seen how both oscillatory integrals of the first and second kind imply sub-level set estimates, and we have also explored the stability of sub-level set estimates.

The aim of this chapter is to explore the possibility of obtaining a *calculus of oscillatory integral estimates* in one dimension for the particular simple example where the derivatives of the phase, at the critical point x_0 , satisfy

$$\Phi'(x_0) = \dots = \Phi^{(k-1)}(x_0) = 0, \quad (4.1.1)$$

while $\Phi^{(k)}(x_0) \neq 0$ with $k \geq 2$. We have seen from Corollary 1.3.3 in Chapter 1 that one obtains for the one dimensional oscillatory integral $\int e^{i\lambda\Phi(x)}\psi(x)dx$ the van der Corput type estimate of $O(\lambda^{-1/k})$, under the hypothesis $\Phi^{(k)}(x_0) \neq 0$

alone without the added condition on the derivatives of the phase as given above in (4.1.1).

However, it is well known, that the extra condition in (4.1.1) allows us to obtain the much stronger full *asymptotic expansion*, in terms of powers of λ , for the oscillatory integral $\int e^{i\lambda\Phi(x)}\psi(x)dx$, where ψ is a smooth function supported in a sufficiently small neighbourhood of x_0 . Under the above hypotheses one obtains the very strong asymptotic conclusion:

$$\int e^{i\lambda\Phi(x)}\psi(x)dx \sim \lambda^{-1/k} \sum_{j=0}^{\infty} a_j \lambda^{-j/k} \quad (4.1.2)$$

in the sense that, for all nonnegative integers N and s , the s -th derivative, with respect to λ , of

$$\int e^{i\lambda\Phi(x)}\psi(x)dx - \lambda^{-1/k} \sum_{j=0}^N a_j \lambda^{-j/k} \quad (4.1.3)$$

is $O(\lambda^{-s-(N+1)/k})$ as $\lambda \rightarrow \infty$.

Note. Each constant a_j that appears in the asymptotic expansion of (4.1.2) depends on only finitely many derivatives of Φ and ψ at x_0 . For example, in the case $k = 2$, we have

$$a_0 = \left(\frac{2\pi}{-i\Phi''(x_0)} \right)^{1/2} \psi(x_0).$$

In n -dimensions, if we have in addition that the critical point x_0 is *nondegenerate*, meaning that the phase Φ satisfies the additional criterion where the $n \times n$ symmetric matrix $\left[\frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right] (x_0)$ is invertible, then we have the following asymptotic expansion:

$$\int_{\mathbf{R}^n} e^{i\lambda\Phi(x)}\psi(x)dx \sim \lambda^{-n/2} \sum_{j=0}^{\infty} a_j \lambda^{-j}$$

as $\lambda \rightarrow \infty$, where the asymptotics hold in the same sense as (4.1.3); and again

each of the constants a_j appearing in the asymptotic expansion above depend on the values of only finitely many derivatives of Φ and ψ at x_0 . Thus, for instance,

$$a_0 = (2\pi)^{n/2} \prod_{j=1}^n (-i\mu_j) \cdot \psi(x_0),$$

where $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of the matrix $\left[\frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right] (x_0)$.

However, the one dimensional setting has the simple advantage that for any smooth phase Φ with an isolated critical point of finite type at x_0 , the corresponding oscillatory integral will have a valid asymptotic formula (4.1.2); and so if one were to consider only oscillatory integral estimates near an isolated critical point, it makes sense to restrict to optimal decay estimates which will in turn imply the full asymptotic formula.

The main exercise then will be for us to identify the conditions under which a phase of the form $\Psi(x) = P(\Phi(x))$, where P and Φ are real-valued and smooth, satisfies $\Psi'(x_0) = \dots = \Psi^{(\ell-1)}(x_0) = 0$ but $\Psi^{(\ell)}(x_0) \neq 0$. The point of doing this is that it then gives us the necessary conditions for establishing a calculus of oscillatory integral estimates in the setting of asymptotic expansions.

4.2 Estimates for oscillatory integrals with polynomial functions of the phase

Apart from the case where $P(t)$ is a monomial, it is very difficult on the sole basis of a measure estimate for the sub-level set of Φ to directly deduce an estimate for the measure of the sub-level set when we have $P(\Phi)$ instead of Φ , where P is an arbitrary polynomial of degree d say. And in general, there is no direct implication from the measure estimate for the sub-level set of Φ to the measure

estimate for the sub-level set of $P(\Phi)$. However, as we have seen in the previous chapter, if our estimate for the sub-level set of Φ comes from an oscillatory integral estimate, then we can use the folklore technique, along with an appropriate decomposition of \mathbf{R} , to by pass this problem in order to obtain an estimate for the sub-level set of $P(\Phi)$.

Unfortunately, we do not have an appropriate hierarchial object for the oscillatory integral as we do for the sub-level set so that we can do the analogous procedure for the oscillatory integral, and thus there is no technique that will do an analogous job for the oscillatory integral as the folklore technique does for the sub-level set. Attempting to utilise sub-level sets bears limited fruit, since even though there are cases where sub-level set estimates do imply oscillatory integral estimates¹, in general this does not always turn out to be the case, as the following example demonstrates very clearly.

Example 4.2.1. Consider $\Phi(x) = x^2 \pm 100$ and $\lambda > 1$, then

$$|\{x \in [0, 1] : |\Phi(x)| < \lambda^{-1}\}| = |\emptyset| = 0$$

and via van der Corput $|\int_0^1 e^{i\lambda\Phi(x)} dx| \leq C\lambda^{-1/2}$.

So, when it comes to the oscillatory integral, we are left with a grim state of affairs. For not only is there no natural way to go from an estimate for $\int e^{i\lambda\Phi(x)}\psi(x)dx$ to an estimate for $\int e^{i\lambda P(\Phi(x))}\psi(x)dx$, and by this, we simply mean to ask the question: if we know that for certain conditions on the phase Φ that we have an estimate of the form

$$\left| \int e^{i\lambda\Phi(x)}\psi(x)dx \right| \leq C\lambda^{-\alpha}$$

for some α and absolute constant C , can we say what the estimate will be if we

¹This was demonstrated in section 1.4.

have $P(\Phi)$ instead of Φ ? There is also no analogous operation to the folklore technique at our disposal that can help us along further in our investigation.

However, even though there is no naturally obvious general method at our disposal, it is still natural to ask if there are any specific situations where one can see that an estimate for $\int e^{i\lambda\Phi(x)}dx$ will imply an estimate for $\int e^{i\lambda P(\Phi(x))}dx$. It turns out that when Φ satisfies the derivative condition given in (4.1.1), then we are able to obtain the asymptotic estimate of the oscillatory integral for when the phase Φ is replaced by $P \circ \Phi$, where P is real-valued and smooth and not just a polynomial function. However, we must stress that this estimate is not uniform over all P as it does depend upon the coefficients of P .

The following result is well known and we simply state it without proof, the interested reader may consult [41] should they wish to see the details.

Proposition 4.2.2. [41] *Suppose $k \geq 2$, and*

$$\Phi'(x_0) = \dots = \Phi^{(k-1)}(x_0) = 0,$$

while $\Phi^{(k)}(x_0) \neq 0$. If ψ is supported in a sufficiently small neighbourhood of x_0 , then

$$\int e^{i\lambda\Phi(x)}\psi(x)dx \sim \lambda^{-1/k} \sum_{j=0}^{\infty} a_j \lambda^{-j/k},$$

in the sense that, for all nonnegative integers N and s ,

$$\left(\frac{d}{d\lambda}\right)^s \left[\int e^{i\lambda\Phi(x)}\psi(x)dx - \lambda^{-1/k} \sum_{j=0}^N a_j \lambda^{-j/k} \right] = O(\lambda^{-s-(N+1)/k}) \quad (4.2.1)$$

as $\lambda \rightarrow \infty$.

We will assume that the phase Φ we are dealing with satisfies the condition on its derivatives given in the above proposition, and we will examine $P \circ \Phi$ for general P real-valued and smooth. For this purpose, we will use the formula of Faà di Bruno which generalises the chain rule to higher derivatives. In fact, the origin, and the development, of Faà di Bruno's formula has a very curious and somewhat controversial history to it; and for an interesting historical account of the formula's development we refer the reader to the recent paper of W. P. Johnson [28], in which there is also given a substantial mathematical treatment of the formula's combinatorial aspects.

Now, for a function given by $\Psi(x) = P(\Phi(x))$, where P and Φ are real-valued and smooth, the Faà di Bruno formula expresses the m -th derivative of Ψ in terms of the derivatives of P and Φ as follows. For $\Psi(x) = P(\Phi(x))$ we have

$$\Psi^{(m)}(x) = \sum_{j=1}^m P^{(j)}(\Phi(x)) \Lambda_{m,j}(\Phi'(x), \dots, \Phi^{(m-j+1)}(x))$$

where $\Lambda_{m,j}$ is the m, j -th Bell polynomial which is defined as

$$\Lambda_{m,j}(r_1, \dots, r_{m-j+1}) = \sum \frac{m!}{\ell_1! \dots \ell_{m-j+1}!} \left(\frac{r_1}{1!}\right)^{\ell_1} \dots \left(\frac{r_{m-j+1}}{(m-j+1)!}\right)^{\ell_{m-j+1}}$$

and the sum defining $\Lambda_{m,j}$ is taken over all tuples $(\ell_1, \dots, \ell_{m-j+1})$ satisfying the constraint conditions $\ell_1 + \dots + \ell_{m-j+1} = j$ and $\ell_1 + 2\ell_2 + \dots + (m-j+1)\ell_{m-j+1} = m$.

We are now ready to prove the following result.

Theorem 4.2.3. *Suppose $k \geq 2$, and*

$$\Phi'(x_0) = \dots = \Phi^{(k-1)}(x_0) = 0,$$

while $\Phi^{(k)}(x_0) \neq 0$. If ψ is supported in a sufficiently small neighbourhood of x_0 , and P is a smooth function which satisfies

$$P'(\Phi(x_0)) = \dots = P^{(n-1)}(\Phi(x_0)) = 0, \quad P^{(n)}(\Phi(x_0)) \neq 0,$$

for $n \geq 1$, then $\Psi(x) = P(\Phi(x))$ satisfies

$$\Psi'(x_0) = \dots = \Psi^{(nk-1)}(x_0) = 0, \quad \Psi^{(nk)}(x_0) \neq 0,$$

and so, we have that

$$\int e^{i\lambda P(\Phi(x))} \psi(x) dx \sim \lambda^{-1/nk} \sum_{j=0}^{\infty} a_j \lambda^{-j/nk},$$

where the asymptotics hold in the same sense as in (4.2.1).

Proof. Our task is to show that $\Psi^{(nk)}(x_0) \neq 0$ and that $\Psi^{(s)}(x_0) = 0$ for every s such that $1 \leq s \leq nk - 1$, the result then follows by applying Proposition 4.2.2.

For each $j \geq 1$, and $m \geq 1$, let us denote by $\Gamma_{m,j}$ the set of tuples $\bar{\ell} = (\ell_1, \dots, \ell_{m-j+1})$ which satisfy the two constraint conditions $\ell_1 + \dots + \ell_{m-j+1} = j$ and $\ell_1 + 2\ell_2 + \dots + (m-j+1)\ell_{m-j+1} = m$. For each $m \geq 1$, We have via the Faà di Bruno formula that

$$\Psi^{(m)}(x_0) = \sum_{j=1}^m P^{(j)}(\Phi(x_0)) \Lambda_{m,j}(\Phi'(x_0), \dots, \Phi^{(m-j+1)}(x_0)) \quad (4.2.2)$$

where $\Lambda_{m,j}$ is of course defined as

$$\Lambda_{m,j}(r_1, \dots, r_{m-j+1}) = \sum_{\bar{\ell} \in \Gamma_{m,j}} \frac{m!}{\ell_1! \dots \ell_{m-j+1}!} \left(\frac{r_1}{1!}\right)^{\ell_1} \dots \left(\frac{r_{m-j+1}}{(m-j+1)!}\right)^{\ell_{m-j+1}}. \quad (4.2.3)$$

For $m = nk$, the strategy will be to show that the only non-zero contribution is

from the term where $j = n$, since from our hypothesis we have that $P^{(n)}(\Phi(x_0))$ is non-zero, and also that all the terms for $1 \leq j \leq n - 1$ in the sum (4.2.2) vanish, and we will see that for $j \geq n + 1$ there is no tuple $(\ell_1, \dots, \ell_{m-j+1})$ in the set $\Gamma_{m,j}$ with $\ell_1 = \dots = \ell_{k-1} = 0$, which in turn implies that for every $j \geq n + 1$ we will have that $\Lambda_{m,j}(\Phi'(x_0), \dots, \Phi^{(m-j+1)}(x_0)) = 0$.

Now when $j = n$ it is easy to see that there is just one tuple $(\ell_1, \dots, \ell_{nk-j+1})$ in the sum defining $\Lambda_{nk,j}(\Phi'(x_0), \dots, \Phi^{(nk-j+1)}(x_0))$ which has $\ell_k = n$ and the remaining ℓ_r 's, with $r \neq k$, equal to zero, otherwise, the first constraint condition is violated. Now, when $j = n$, we have

$$\Gamma_{nk,j} = \{\bar{\ell} : \ell_k = n\} \cup \{\bar{\ell} : 0 \leq \ell_k \leq n - 1\},$$

and we claim that for every $(\ell_1, \dots, \ell_{nk-j+1}) \in \{\bar{\ell} : 0 \leq \ell_k \leq n - 1\}$ it cannot be true that $\ell_1 = \dots = \ell_{k-1} = 0$. We prove the claim by arguing by contradiction.

Let us suppose then that for every tuple in the set $\{\bar{\ell} : 0 \leq \ell_k \leq n - 1\}$ it is true that $\ell_1 = \dots = \ell_{k-1} = 0$. Now, for any tuple belonging to the set $\{\bar{\ell} : 0 \leq \ell_k \leq n - 1\}$ we have that $\ell_k = n - p$ for some p such that $1 \leq p \leq n$. Hence, applying this together with our hypothesis that $\ell_1 = \dots = \ell_{k-1} = 0$, we obtain from the first constraint condition that

$$\ell_{k+1} + \dots + \ell_{nk-n+1} = p. \quad (4.2.4)$$

In exactly the same way, we obtain from the second constraint condition that

$$(k+1)\ell_{k+1} + \dots + (nk - n + 1)\ell_{nk-n+1} = pk. \quad (4.2.5)$$

We then have that

$$\begin{aligned}
pk &= (k+1)\ell_{k+1} + \dots + (nk-n+1)\ell_{nk-n+1} \\
&\geq (k+1)(\ell_{k+1} + \dots + \ell_{nk-n+1}) \\
&= (k+1)p
\end{aligned}$$

where the last equality follows by using (4.2.4), and hence we arrive at the contradiction that $p \leq 0$. We therefore conclude that, for every tuple $\bar{\ell}$ in the set $\{\bar{\ell} : 0 \leq \ell_k \leq n-1\}$, the terms in the sum $\Lambda_{nk,n}(\Phi'(x_0), \dots, \Phi^{(nk-n+1)}(x_0))$ which correspond to all such tuples will be zero, and so,

$$\Lambda_{nk,n}(\Phi'(x_0), \dots, \Phi^{(nk-n+1)}(x_0)) = \frac{(nk)!}{n!} (\Phi^k(x_0))^n \neq 0.$$

Finally, we claim that for all $j \geq n+1$ there is no tuple $(\ell_1, \dots, \ell_{nk-j+1})$, with $\ell_1 = \dots = \ell_{k-1} = 0$, belonging to the set $\Gamma_{nk,j}$, and thus that for all $j \geq n+1$ we have $\Lambda_{nk,j}(\Phi'(x_0), \dots, \Phi^{(nk-j+1)}(x_0)) = 0$. Again we prove the claim by arguing by contradiction. The proof will run in a very similar way to the one just given above, and so we will economise on the details. Moreover, the argument for $m = nk$ works for general m in the range $m \leq nk$, and not just for the case $m = nk$, and so, we will do it for general m in this range.

In a similar way as before we write

$$\Gamma_{m,j} = \{\bar{\ell} : \ell_k = j\} \cup \{\bar{\ell} : 0 \leq \ell_k \leq j-1\}.$$

We claim that the first set is empty, and that in every tuple belonging to the second set we must have $\ell_r \neq 0$ for some r such that $1 \leq r \leq k-1$. The first claim is true, for if it was the case that the set $\{\bar{\ell} : \ell_k = j\}$ was non-empty then by the first constraint condition the only tuple that could belong to it would be

the one which has $\ell_k = j$ and the remaining ℓ_r 's, with $r \neq k$, equal to zero. The second constraint condition then implies that $m = kj$, but since $j \geq n + 1$ and $m \leq nk$ we obtain a contradiction.

The second claim follows by exactly the same argument as before. We suppose that in every tuple belonging to the set $\{\bar{\ell} : 0 \leq \ell_k \leq j - 1\}$ it is true that $\ell_1 = \dots = \ell_{k-1} = 0$. Now, for any tuple belonging to the set $\{\bar{\ell} : 0 \leq \ell_k \leq j - 1\}$ we have that $\ell_k = n - j$ for some p such that $1 \leq p \leq j$. Hence, applying this, along with our hypothesis that $\ell_1 = \dots = \ell_{k-1} = 0$, in both of the constraint conditions, we arrive at the inequality

$$m - jk \geq p,$$

and since $j \geq n + 1$ implies that $jk \geq nk + k > m$, we arrive at the contradiction that $p < 0$.

We next show that for every m in the range $1 \leq m \leq nk - 1$ that $\Psi^{(m)}(x_0) = 0$.

Observe that since

$$\{m : 1 \leq m \leq nk - 1\} = \{m : 1 \leq m \leq n - 1\} \cup \{m : m = n\} \cup \{m : n + 1 \leq m \leq nk - 1\}$$

there are thus three cases to consider. In the first case, when m lies in range $1 \leq m \leq n - 1$, all the terms for $1 \leq j \leq m$ in the sum (4.2.2) vanish by hypothesis. In the second case, when $m = n$, we have

$$\Psi^{(m)}(x_0) = \sum_{j=1}^{n-1} P^{(j)}(\Phi(x_0)) \Lambda_{m,j}(\Phi'(x_0), \dots, \Phi^{(m-j+1)}(x_0)) + P^{(n)}(\Phi(x_0)) \Lambda_{n,n}(\Phi'(x_0)).$$

The first term in the sum above vanishes again by hypothesis, and the second term vanishes since $\Lambda_{n,n}(\Phi'(x_0)) = (\Phi'(x_0))^n$. Finally, in the third case, when m is in the range $n+1 \leq m \leq nk-1$, we split the sum in (4.2.2) as

$$\sum_{j=1}^m = \sum_{1 \leq j \leq n-1} + \sum_{j=n} + \sum_{n+1 \leq j \leq m}.$$

We have seen already that first two sums vanish. Moreover, the third sum also vanishes on account of our earlier observation that for every $j \geq n+1$, and any $m \leq nk$, we have $\Lambda_{m,j}(\Phi'(x_0), \dots, \Phi^{(m-j+1)}(x_0)) = 0$.

□

As a simple corollary we can now illustrate the simple case of when $P(t) = t^n$, $n \geq 2$, is a monomial function.

Corollary 4.2.4. *Suppose $k \geq 2$, and*

$$\Phi'(x_0) = \dots = \Phi^{(k-1)}(x_0) = 0,$$

while $\Phi^{(k)}(x_0) \neq 0$. If ψ is supported in a sufficiently small neighbourhood of x_0 , then

$$\int e^{i\lambda[\Phi(x)]^n} \psi(x) dx \sim \begin{cases} \lambda^{-1/nk} \sum_{j=0}^{\infty} a_j \lambda^{-j/nk} & \Phi(x_0) = 0, \\ \lambda^{-1/k} \sum_{j=0}^{\infty} a_j \lambda^{-j/k} & \Phi(x_0) \neq 0, \end{cases}$$

where the asymptotics hold in the same sense as in (4.2.1), and $n \geq 1$.

Proof. For $n = 1$ we obtain the same asymptotic result as given in Proposition 4.2.2 whether $\Phi(x_0) = 0$ or $\Phi(x_0) \neq 0$. However, if $n \geq 2$, and if $\Phi(x_0) = 0$, then since $P(t) = t^n$, it is clear that we have

$$P'(\Phi(x_0)) = \dots = P^{(n-1)}(\Phi(x_0)) = 0,$$

while $P^{(n)}(\Phi(x_0)) \neq 0$, and so, the first asymptotic estimate follows by application of Theorem 4.2.3. If $\Phi(x_0) \neq 0$ then it is also clear that $P'(\Phi(x_0)) \neq 0$, and so, by applying Theorem 4.2.3 the second asymptotic estimate is obtained instead. \square

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